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FINAL REPORT

AN INVESTIGATION FOR THE IMPLEMENTATION OF ADAPTIVE GUIDANCE

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ABSTRACT

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This report summarizes investigations which considered the trajectories resulting from the application of a small thrust to an object in an inverse square central force field. Analytical solutions for the special cases involving radial, normal, circumferential, and tangential thrusting are reviewed and extended. A second-order perturbation theory is derived for a vehicle departing from a circular orbit. The trajectory is produced by a thrust vector maintained at a constant, but arbitrarily chosen, angle with respect to the radius vector. Numerical results of the second-order theory are presented which show representative low-thrust trajectories. The perturbation theory is extended to accept elliptical orbit starting conditions.

author

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ILLUSTRATIONS

SYMBOLS

r	=	orbit radius = distances from center of attraction to the vehicle
r_o	=	initial orbit radius
ρ	=	dimensionless orbit radius
t	=	time measured from initial position
τ	=	dimensionless time
θ	=	polar angle measured from initial position
ϕ	=	angle between the radius vector and the instantaneous velocity vector.
ψ	=	angle between the radius vector and the thrust vector
s	=	dimensionless arc length
α	=	thrust acceleration $\div$ initial gravitational acceleration
f	=	thrusting force
M	=	mean anomaly
n_o	=	mean motion of an initially circular orbit
e	=	eccentricity
e^x	=	$\exp[x]$, the exponential function
T	=	kinetic energy
V	=	velocity
E	=	instantaneous energy
h	=	instantaneous angular momentum
i	=	$\sqrt{-1}$
u	=	$1/\rho$
$0(\alpha)$	=	the neglected terms of an infinite series in which the coefficients of the neglected terms have positive powers of α as factors
	=	a derivative with respect to time when the dot is over a variable
$\Delta\rho$	=	deviation from the reference circular orbit radius

SECTION 1

INTRODUCTION

This report summarizes investigations performed at Aeronutronic under Contract NAS 8-5248, during the period from March 23, 1963 to March 23, 1964. The subject of these investigations has been the trajectories resulting from the application of a small thrust to an object in an inverse square central force field. Interest in low-thrust trajectories has been increasing in parallel with development efforts to accomplish workable low-thrust electrical propulsion systems. When the propulsion technology has sufficiently evolved, low thrust devices will find application to such diverse space missions as interplanetary transfers, station keeping and attitude control of geocentric satellites, and enlargement or modification of geocentric satellite orbits.

Analysis of low-thrust trajectories has, for the most part, been concentrated in areas best suited to interplanetary transfers. For such transfers, the ratio of thrust acceleration to gravitational acceleration is such that the trajectory involves a heliocentric arc of less than one revolution.

Two general approaches have been employed by various investigators in examining such trajectories. One approach (historically the first) has been to evaluate, by analytical techniques, the trajectories resulting from the application of a thrust acceleration which obeys a preselected thrust direction program. Thrust programs considered for planar problems have generally consisted of thrusting along the flight path, normal to the flight path, along the radius vector, or normal to the radius vector. Of all these possibilities, only the case of radial thrusting has yielded closed form analytical solutions. The other thrust programs have been treated approximately by asymptotic techniques or perturbation theory.

The second general approach to the low thrust trajectory problem has involved the determination of optimal thrust programs using the calculus of variations. These efforts are extremely valuable, but are hampered by analytical complexities which necessitate the use of numerical techniques for all but the most trivial cases. These problems are further compounded when trajectories in a strong gravitational field are considered. Such trajectories (in geocentric space, for example) may involve many revolutions about the central body, making the use of ordinary numerical integration techniques quite expensive for a given computational accuracy. In addition, the absence of convenient approximate analytical solutions hampers the preliminary analysis of such trajectories. These problems supplied the primary motivation for the present study.

The approach employed in the present study belongs in the first category of analysis described above, i.e., trajectory descriptions for specified thrust programs are sought. In Section 2, analytical solutions for thrust programs involving radial, normal, circumferential, and tangential thrusting are reviewed and extended. In Section 3, a second-order perturbation solution is presented which permits thrusting to occur at arbitrary constant angles with respect to the radius vector. Sample numerical results are presented, and the application of the theory to transfers between circular orbits and to trajectories involving multiple revolutions is discussed.

SECTION 2

LOW-THRUST TRAJECTORIES SPECIAL CASES

This section contains approximate expressions which dictate the motion of a vehicle under small constant radial, normal, tangential, and circumferential thrust accelerations. The vehicle under consideration is initially moving in a circular orbit in an attractive inverse-square, central force field.

2.1 APPROXIMATE SOLUTION FOR LOW RADIAL THRUST

A particle is moving in a circular Kepler orbit when, at time $t = t_o$, a constant radial force (thrust) is added. The resulting motion is exactly described in terms of elliptic integrals of the first, second, and third kinds. Copeland[1], with corrections by Karrenberg[2] and Au[3], has carried out the analysis for the four classes of motion: $\alpha < 0$, $0 < \alpha < \frac{1}{8}$, $\alpha = \frac{1}{8}$, and $\alpha > \frac{1}{8}$, where α is the ratio of thrust acceleration to the initial gravitational acceleration. The value $\alpha = \frac{1}{8}$ is significant because it corresponds to the minimum continuous thrust needed to escape.

The system has an energy integral and a momentum integral which may be used to define the region of motion as a function of α. In particular, the motion is periodic when $\alpha < \frac{1}{8}$. The region of motion and periodicity are reviewed in Section 2.1.1.

The intention here is to derive an approximate analytical description of the motion, not involving elliptic integrals, when the thrust is numerically small, that is, when $|\alpha| << \frac{1}{8}$. As Levin[4] points out, the exact solution in terms of elliptic integrals are not particularly convenient to use.

When the thrust is small, the departure from the initial Kepler orbit may also be expected to be small. Lass and Lorell[5] using the method of Kryloff and Bogoliuboff, and Citron[6] using the method of variation of parameters have determined the first order changes in the Kepler constants due to a small radial thrust. It is important to know that the Kepler orbit itself was used as the zero order approximation in these studies. Lass and Lorell start with an elliptical Kepler orbit and determine the changes in the constants over a complete revolution. However, Citron starts with a circular Kepler orbit and determines the instantaneous changes in the constants.

Section 2.1.2 shows that, with respect to a certain pseudo-angle ϕ, the orbit is an ellipse with eccentricity $|\alpha|$ and mean motion $(1-3\alpha)\ n_o$. In real space the orbit is quasi-elliptical since the apsidal angle is $(1+\alpha)\ \pi$ rather than π.

2.1.1 REGIONS OF MOTION

A particle of unit mass is acted upon by an attractive inverse square force $-c/r^2$ and a constant radial force f. The kinetic and potential energies in plane polar coordinates are

$$T = \frac{1}{2}\left[\dot{r}^2 + r^2\ \dot{\theta}^2\right], \qquad V = -\frac{c}{r} - fr$$

respectively. Since neither T nor V depend explicitly on the time t, their sum K is a constant. Also, since neither T nor V are explicit functions of θ the momentum

$$h = r^2\ \dot{\theta} \tag{1}$$

is a constant. The energy integral may be expressed as

$$\left[\frac{dr}{d\theta}\right]^2 = \left[\frac{r^2}{h}\right]^2 \left\{ 2\left(K + \frac{c}{r} + fr\right) - \left[\frac{h}{r}\right]^2 \right\} . \tag{2}$$

Dividing by r^4 and using the transformation $u = 1/r$, which is familiar in the solution of the Kepler problem, equation (2) becomes

$$\left[\frac{du}{d\theta}\right]^2 = \frac{2}{h^2}\left(K + cu + \frac{f}{u}\right) - u^2 . \tag{3}$$

If f were zero and if the particle were moving in a circular Kepler orbit $u = u_o$, then

$$E_o = -\frac{cu_o}{2} , \qquad h^2 = \frac{c}{u_o} .$$

At time $t = t_o$ a force f is added and a new energy constant K is evaluated, then

$$K = -\frac{cu_o}{2} - \frac{f}{u_o} .$$

For these initial conditions equation (3) becomes

$$\left[\frac{du}{d\theta}\right]^2 = -(u_o - u)^2 + \frac{2f}{c}\left[\frac{u_o}{u} - 1\right] \tag{4}$$

Dividing by u_o^2, to make equation (4) nondimensional, and setting

$$\lambda = \frac{u}{u_o} , \qquad \alpha = \frac{f}{cu_o^2}$$

equation (4) becomes

$$\left[\frac{d\lambda}{d\theta}\right]^2 = -(1-\lambda)^2 + 2\alpha\lambda^{-1}(1-\lambda) \quad . \tag{5}$$

Consider the roots for the equation

$$F(\lambda) = -(1-\lambda)^2 + 2\alpha\lambda^{-1}(1-\lambda) = 0.$$

When $\alpha = 0$, $F(\lambda)$, is quadratic with two equal roots $\lambda = 1$. When $\alpha \neq 0$, $F(\lambda)$, is cubic with one real root $\lambda = 1$. The other two roots satisfy the equation

$$\lambda^2 - \lambda + 2\alpha = 0$$

which has the roots

$$\frac{1}{2}\left[1 \pm \sqrt{1 - 8\alpha}\right].$$

For $|\alpha| << \frac{1}{8}$ the roots are approximately $1 - 2\alpha$ and 2α. When $\alpha = \frac{1}{8}$, there are two equal roots at $\lambda = \frac{1}{2}$. When $\alpha < \frac{1}{8}$, but not equal to zero, these roots are real and unequal, and when $\alpha > \frac{1}{8}$, these roots are complex. The function $F = F(\lambda)$, is sketched in Figure 1. For $\alpha < \frac{1}{8}$ the motion is periodic with limits $\lambda = 1$ and $\lambda = \frac{1}{2}[1 + \sqrt{1 - 8\alpha}]$. The value $\alpha = \frac{1}{8}$ corresponds to the minimum value for λ to approach zero, which implies escape.

With $h^2 = c/u_o$ and $n_o^2 = cu_o^3$, equation (1) may be written as

$$\dot{\theta} = n_o\lambda^2 \quad . \tag{6}$$

Clearly, $\dot{\theta} > 0$ means that λ is a single-valued function of θ over the interval $0 \leq \theta \leq 2\pi$.

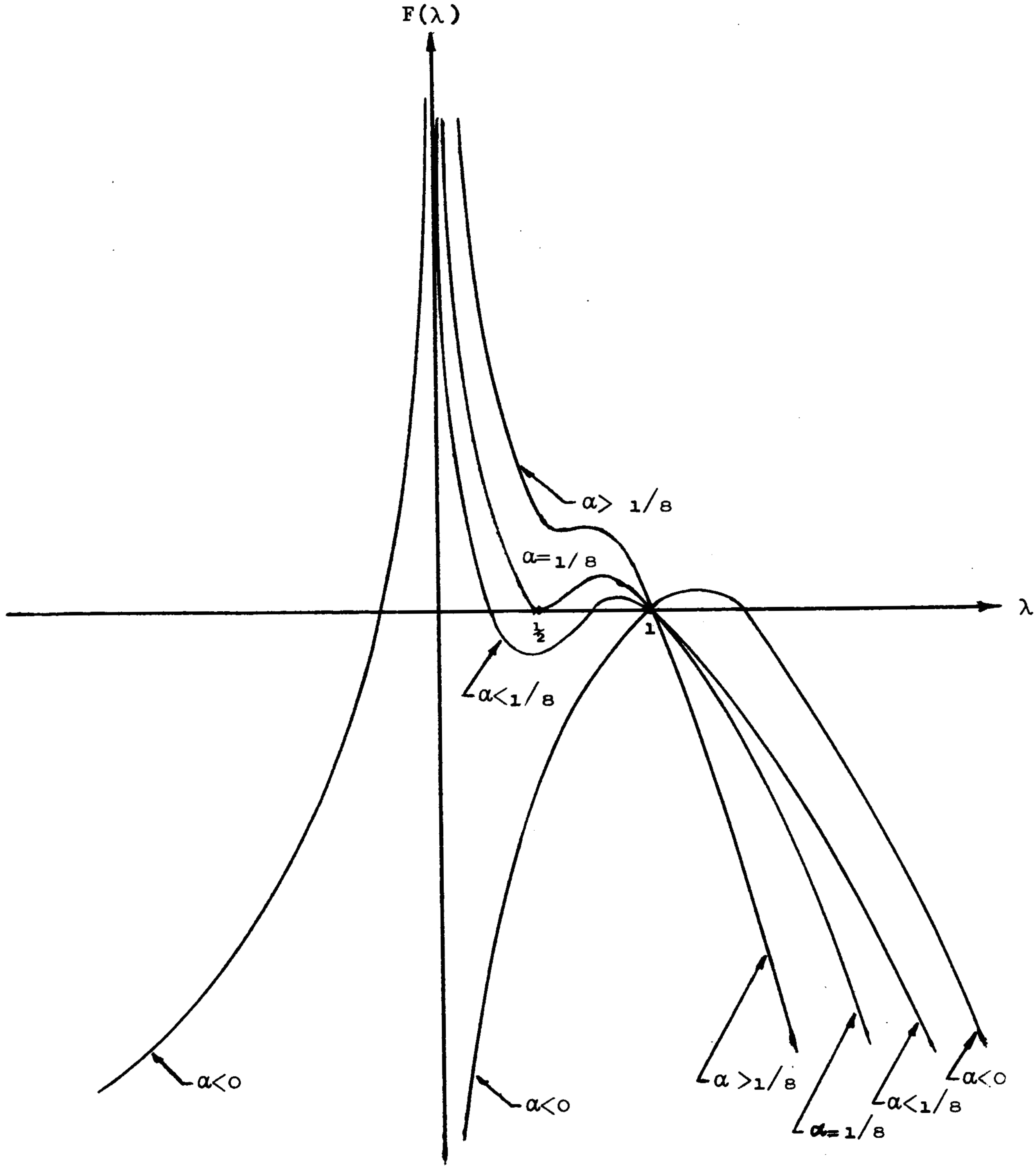


FIGURE 1. RADIAL THRUST FUNCTION F(λ)

The energy integral may be also written as

$$E - fr = K$$

where E is the instaneous energy of the Kepler ellipse, which the particle would follow if f were set equal to zero at any instant of time. In terms of the initial conditions

$$\frac{E}{E_o} = 1 - 2\alpha\lambda^{-1}(1-\lambda) \quad . \tag{7}$$

The instantaneous values for the semi-major axis, a, and the eccentricity, e, for the Kepler orbit are given by

$$\frac{a}{a_o} = \frac{E_o}{E}, \qquad e = \sqrt{1 - \frac{E}{E_o}} \quad .$$

2.1.2 AN APPROXIMATE SOLUTION WHEN $|\alpha| << \frac{1}{8}$

Equation (5) is a first integral of the equation

$$\frac{d^2\eta}{d\theta^2} + \eta = - \frac{\alpha}{(1+\eta)^2} \quad . \tag{8}$$

where $\eta = \lambda - 1$. For $|\alpha| << \frac{1}{8}$, then $-2\alpha \leq \eta \leq 0$ in which case

$$\frac{d^2\eta}{d\theta^2} + (1 - 2\alpha)\eta = -\alpha \tag{9}$$

is an excellent approximation to equation (8). Setting

$$\phi = (1-\alpha)\theta \tag{10}$$

and dropping terms which contain α^2 equation (9) becomes

$$\frac{d^2\eta}{d\,\phi^2} + \eta = -\,\alpha \tag{11}$$

The solution to equation (11) is

$$\eta(\phi) = -\,\alpha + A\cos(\phi - W),\ \delta(\phi) = -\,A\sin(\phi - W) \tag{12}$$

where $\delta(\phi) \equiv \frac{d\eta}{d\phi}$,

and A and W are constants of integration.
For the initial conditions $\eta(0) = 0$ and $\delta(0) = 0$, then $A = \alpha$ and $W = 0$. With respect to ϕ, the orbit

$$\lambda(\phi) = 1 - \alpha(1 - \cos\phi) \tag{13}$$

is an ellipse with eccentricity α. However, with respect to θ the orbit is quasi-elliptical, that is, the apsidal angle is $(1 + \alpha)\pi$ rather than π. The orbit geometry is sketched in Figure 2a for $\alpha > 0$ and in Figure 2b for $\alpha > 0$.

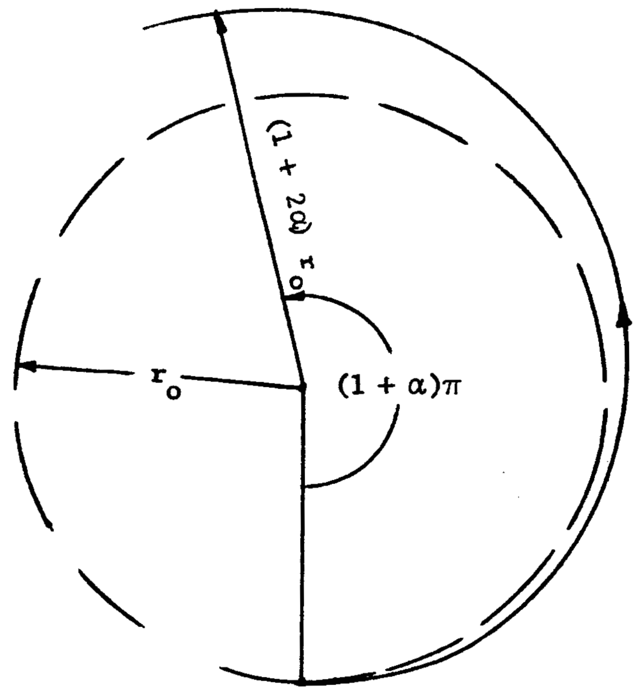
The time along the orbit is determined by substituting equation (113) into equation (6) and making use of equation (10):

$$\dot{\phi} = (1 - \alpha)\, n_o \,[1 - \quad (1 - \cos\phi)]^2$$

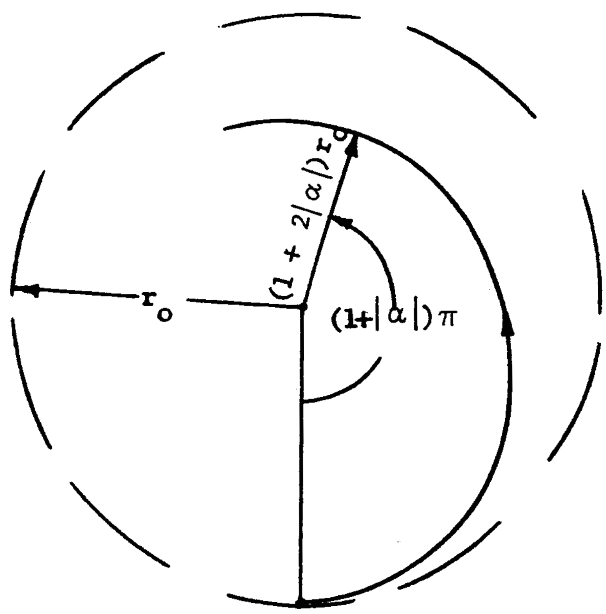
$$\approx (1 - 3\alpha)\, n_o\, (1 + 2\alpha\cos\phi)\ . \tag{14}$$

The solution to equation (14) is

$$\phi - 2\,\alpha\sin\phi = (1 - 3\alpha)\, n_o\, (t - t_o) \equiv M \tag{15}$$



a) $\alpha > 0$



b) $\alpha < 0$

FIGURE 2. RADIAL THRUST TRAJECTORIES

where t_o is the constant of integration. Since terms of order α^2 have been ignored, ϕ may be interpreted as either the eccentric or the true anomaly. When $\alpha > 0$, ϕ is measured from the radius vector for the near apse. Conversely, when $\alpha < 0$, ϕ is measured from the radius vector for the far apse. It is significant that the mean motion $(1 - 3\alpha)\, n_o$ predicted in equation (15) is different than the mean motion n_o for the initial Kepler orbit.

The substitution of equation (13) into equation (7) yields

$$\frac{E}{E_o} = 1 - 2\alpha^2 (1 - \cos\phi) \tag{16}$$

The instantaneous values for a and e for the Kepler ellipse become:

$$\frac{a}{a_o} = 1 + 2\alpha^2 (1 - \cos\phi) \ , \tag{17}$$

$$e = + |\alpha| \sqrt{2(1 - \cos\phi)} \ . \tag{18}$$

The average value for e during one cycle $\phi = 0$ to 2π is

$$\langle e \rangle = \frac{|\alpha|}{2\pi} \int_0^{2\pi} \sqrt{2(1 - \cos\phi)}\, d\phi = \frac{|\alpha|}{2\pi} \int_0^{2\pi} \sin\frac{\phi}{2}\, d\phi = \frac{4|\alpha|}{\pi} \ . \tag{19}$$

The expressions (18) and (19) are different than the corresponding expressions derived by Citron[6]. The difference arises from the fact that Citron treated the semi-major axis, a, as a constant when he integrated to obtain the eccentricity, e. However, a is a periodic function of ϕ as can be seen from equation (17).

2.2 APPROXIMATE SOLUTION FOR LOW NORMAL THRUST

A particle is moving in an initially circular Kepler orbit when a constant force is applied perpendicular to the instantaneous velocity vector and in the plane of motion. The resulting motion of the particle under this normal force was studied by Rodriguez[7] who revealed the possibility of reducing the normal case to quadratures. The complete solution is developed in the following pages for the entire range of normal force. Since the applied force is perpendicular to the velocity, the energy is conserved. Consequently, the semi-major axis of the instantaneous Kepler ellipse (the path which would be traced by a particle if the normal force were removed) is equal to the radius of the initial circular orbit. The particle can never move farther than twice the radius of the initial orbit from the point of central attraction.

The energy integral may be used to reduce the fourth-order system of equation, which describe the motion, by two orders. A complete reduction to quadratures is possible when the problem is formulated in plane polar coordinates but, unlike the radial case, the quadratures are not tabulated integrals. However, even without evaluating the quadratures, the totality of motions can be determined. The first step in this direction is to determine the angular momentum as an explicit function of the distance from the point of central attraction.

2.2.1 REDUCTION TO QUADRATURES

Let O be the point of central attraction and P be the particle. The constant normal force per unit of mass f is positive in the direction shown in Figure 3. The angle between the radial direction and the instantaneous velocity vector V is ϕ. The unit of mass is selected so that the universal constant of gravitation is unity. The energy and angular momentum are given by

$$E = \frac{1}{2} V^2 - \frac{1}{r} , \qquad h = r^2 \dot{\theta}$$

respectively. E is constant but h is not. Its time derivative is given by

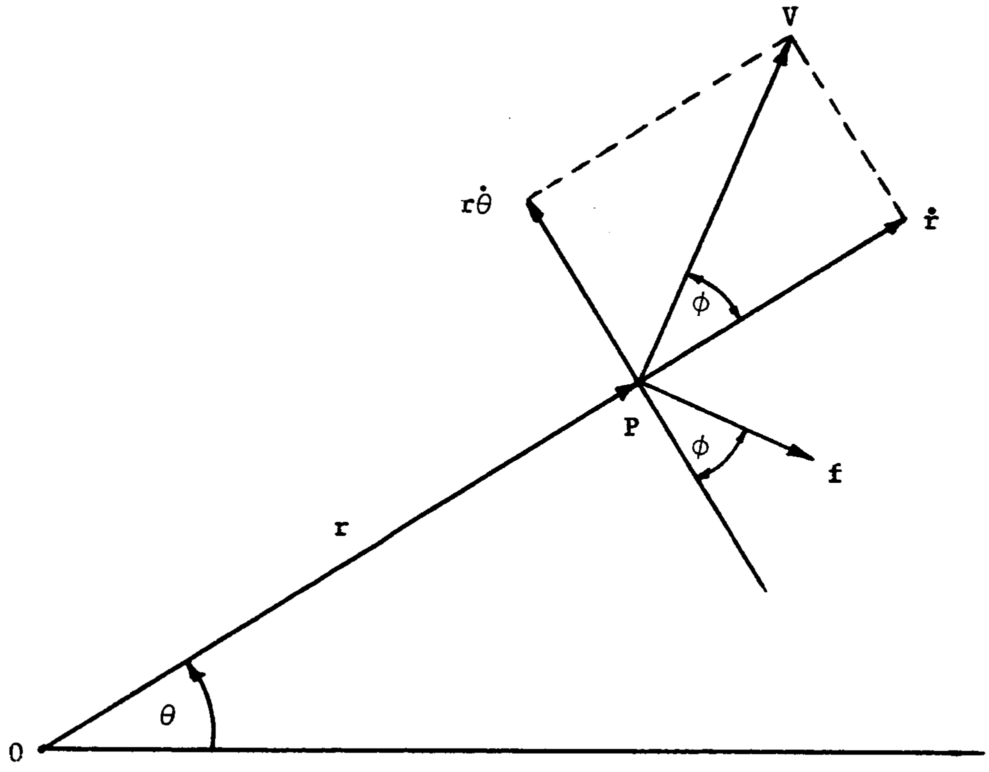


FIGURE 3. NORMAL THRUST GEOMETRY

$$\dot{h} = - fr \cos \phi = - fr \dot{r}/V .$$

Using the energy integral to eliminate V, the differential form

$$dh = - fr \left[2(E + \frac{1}{r}) \right]^{-\frac{1}{2}} dr$$

is obtained. Integration in the case when E is negative (corresponding to a Kepler ellipse) provides

$$h = \frac{-f}{2E\sqrt{2}} \left\{ r^2 (E + \frac{1}{r})^{1/2} (1 - \frac{3}{2Er}) - \frac{3}{2E} (-E)^{-1/2} \tan^{-1} \left[(E + \frac{1}{r}) / -E \right]^{1/2} \right\} + C$$

where C is the constant of integration.

The following analysis is based on the condition that the initial Kepler orbit is circular with radius r_o. E and the initial value of h are given by

$$E = - \frac{1}{2r_o} , \qquad h_o = r_o^{1/2}$$

With the nondimensional parameters

$$\rho = \frac{r}{r_o} , \qquad \alpha = f r_o^2$$

the constant of integration becomes

$$C = \sqrt{r_o} \left[1 - \alpha (2 + \frac{3\pi}{4}) \right]$$

and

$$h = \sqrt{r_o} \left[1 - \alpha U(\rho) \right] \tag{20}$$

The time quadrature

$$\tau = \int \rho \left[2\rho - \rho^2 - (1 - \alpha U)^2 \right]^{-1/2} d\rho \ + \ \text{constant} \tag{22}$$

is analogous to the Kepler equation for the Kepler problem.

The differential equation for the orbit

$$\left[\frac{d\rho}{d\theta} \right]^2 = \left[\frac{\rho}{1 - \alpha U} \right]^2 \left[2\rho - \rho^2 - (1 - \alpha U)^2 \right] \tag{23}$$

is obtained from equation (21) by the operation

$$\frac{d\rho}{d\theta} = \frac{d\rho}{d\tau} \frac{d\tau}{d\theta}$$

where

$$\frac{d\theta}{d\tau} = \frac{1}{\rho^2} (1 - \alpha U) \tag{24}$$

which follows from the definition $h = r^2 \dot{\theta}$.

The orbit equation is also reducible to quadrature:

$$\theta = \int \frac{1 - \alpha U}{\rho} \left[2\rho - \rho^2 - (1 - \alpha U)^2 \right]^{-1/2} d\rho \ + \text{constant} \tag{25}$$

The quadrature equations (22) and (25) appear to be intractable. Nevertheless, a complete qualitative description of the motion can be obtained without carrying out the integration.

2.2.2 REGIONS OF MOTION

From equation (21), the motion is imaginary if

$$f\ (\rho) = - (\rho - 1)^2 + 2\alpha U - \alpha^2 U^2$$

is negative. A necessary (though not sufficient) condition for the motion to be real is that αU be positive. An examination of the sign of U shows that, if the normal force is directed initially outward (inward), the trajectory will never move interior (exterior) to the initial circular orbit.

The question arises whether or not $f(\rho)$ vanishes at any other value $\rho = a$ besides 1. Certainly a would depend on α. The two roots for α which satisfy the equation $f(\rho) = 0$ are

$$\alpha(\rho) = \frac{1}{U}\left[1 \pm (2\rho - \rho^2)^{1/2}\right] \tag{26}$$

from which

$$\alpha(0) = \left[2 - \frac{3\pi}{4}\right]^{-1} \approx -2.809$$

$$\alpha(1) = \pm \infty$$

$$\alpha(2) = \left[2 + \frac{3\pi}{4}\right]^{-1} \approx 0.230$$

The values for α which satisfy equation (26) are shown as functions of ρ in Figure 4. The solid curve in Figure 4 occurs when the minus sign is taken in equation (26), while the two dashed curves occur when the plus sign is used. The value of ρ along these curves is denoted by a. For any given value of α the function $f(\rho)$ is positive and the motion is real in the region between $\rho = 1$ and $\rho = a$. Exterior regions are inaccessible and are so labeled in Figure 4.

It is significant that a unique value $\alpha(0)$ is required to reach the origin $\rho = 0$ and that a second unique value $\alpha(2)$ is required to reach the outer limit $x = 2$. Indeed, it is logical to expect the motion to be qualitatively different in the regions: $\alpha < \alpha(0)$, $\alpha(0) < \alpha < 0$, $0 < \alpha < \alpha(2)$, and $\alpha > \alpha(2)$.

2.2.3 QUALITATIVE DESCRIPTION OF THE MOTION

Considerable information can be obtained by examining equation (21):

$$\left[\frac{d\rho}{d\tau}\right]^2 = R(\rho), \qquad R(\rho) = \frac{1}{\rho^2}\left[2\rho - \rho^2 - (1 - \alpha U)^2\right]$$

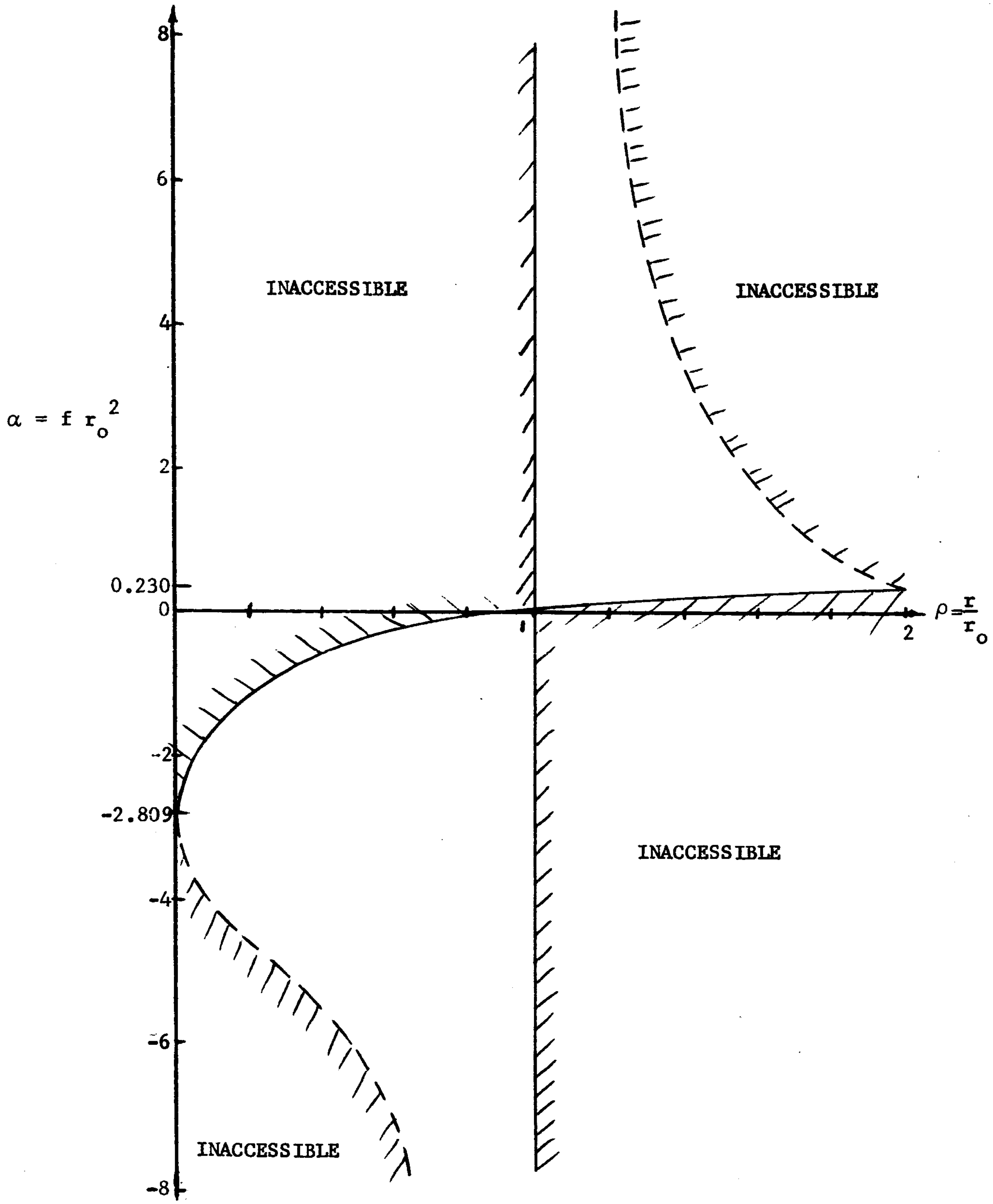


FIGURE 4. REGIONS OF MOTION

$R(\rho)$ has the following properties:

(1) It is continuous.

(2) It is zero at $\rho = 1$ and $\rho = a$. The only exception occurs when $a = 0$.

(3) It is positive in the region between $\rho = 1$ and $\rho = a$.

(4) $dR/d\rho$ does not vanish at $\rho = 1$ and $\rho = a$.

Consequently, the trajectory $\rho = \rho(\tau)$ has the following characteristics:

(1) $\rho(\tau)$ lies between $\rho = 1$ and $\rho = a$ for all values of τ.

(2) $d\rho/d\tau$ only vanishes at $\rho = 1$ and $\rho = a$. However, at $a = 0$ the derivative does not exist.

(3) $\rho(\tau) = \rho(-\tau)$ when the origin $\tau = 0$ is taken at $\rho = 1$ or $\rho = a$.

(4) $\rho(\tau)$ is periodic with period 2K, that is, $\rho(\tau) = \rho(\tau + 2K)$.

The trajectories have the forms shown in Figure 5. It remains to establish the forms for the orbits, $\rho = \rho(\theta)$.

The direction of motion along the boundary $\rho = 1$ is direct. In fact $d\theta/d\tau = 1$. The direction of motion along the boundary $\rho = a$ can be established from equation (24) which provides:

$d\theta/d\tau > 0$	for $\alpha(0) < \alpha < \alpha(2)$
$d\theta/d\tau < 0$	for $\alpha < \alpha(0)$ and $\alpha > \alpha(2)$
$d\theta/d\tau = 0$	at $\alpha = \alpha(2)$
$d\theta/d\tau = +\infty$	as α approaches $\alpha(0)$ from the positive side
$d\theta/d\tau = -\infty$	as α approaches $\alpha(0)$ from the negative side

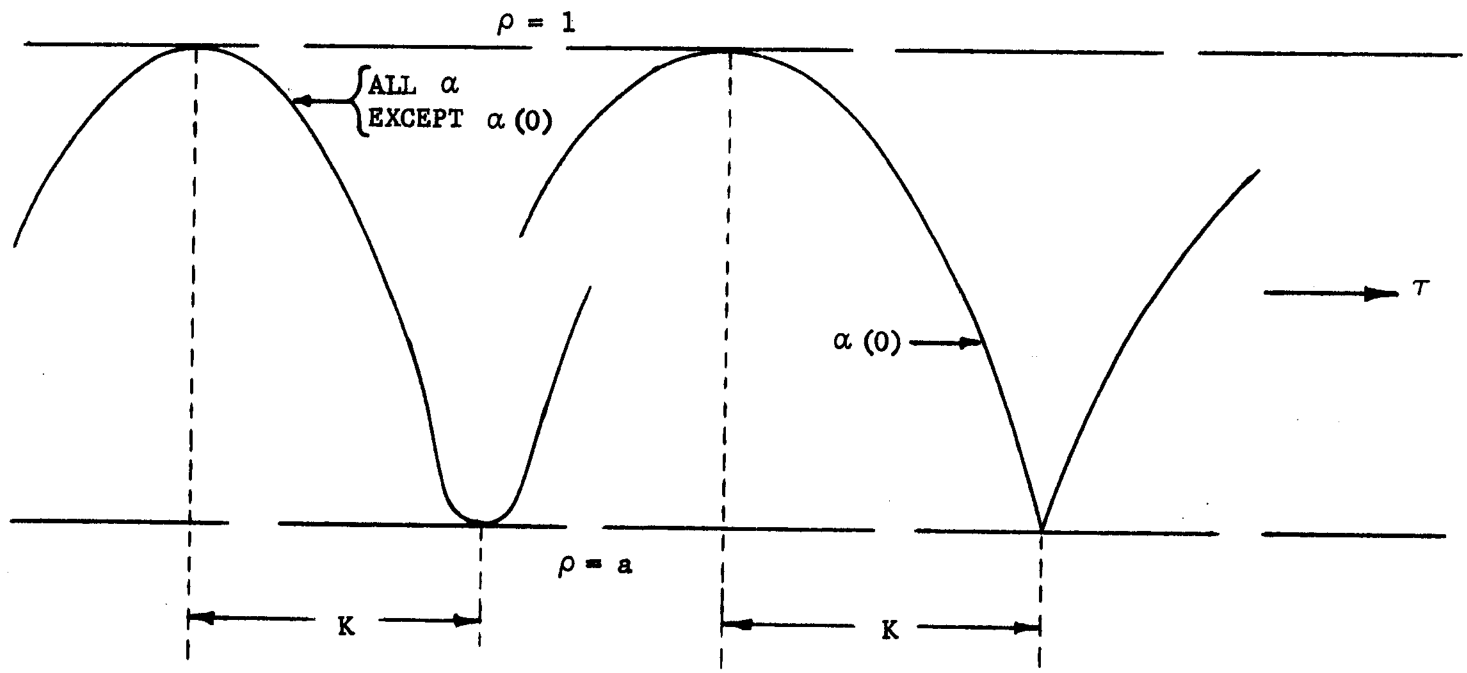


FIGURE 5. TRAJECTORIES $\rho = \rho(\tau)$

An examination of equation (23) shows that $d\rho/d\theta$ vanishes at $\rho = 0$ and $\rho = a$ except at $a = 2$ where $d\rho/d\theta$ does not exist.

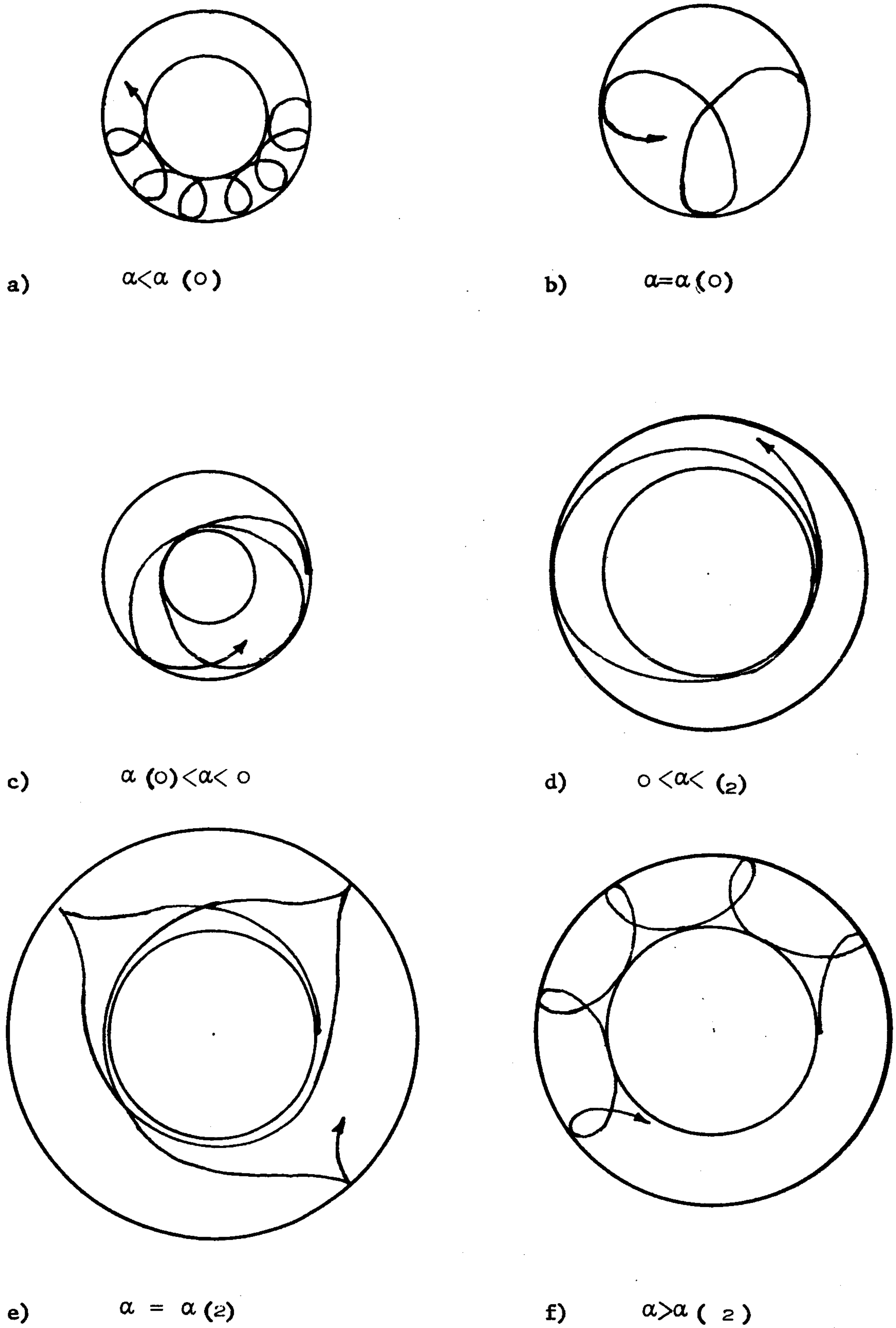
A complete picture (Figure 6) of the orbits can now be formed. ρ is periodic in τ and θ, however, the orbits themselves are not in general periodic since they do not close. Indeed, the only periodic orbits which do exist are isolated. The sign of $d\theta/d\tau$ determines whether the motion is direct or rectrograde. For $\alpha = \alpha(2)$ the orbits have cusps at the outer boundary and for $\alpha = \alpha(0)$ the orbits pass through zero.

2.3 APPROXIMATE SOLUTION FOR LOW TANGENTIAL OR CIRCUMFERENTIAL THRUST USING THE ASYMPTOTIC METHOD

Consider the motion of a particle, initially in a circular Kepler orbit, with constant thrust acceleration in either the tangential (along the velocity vector) or circumferential (perpendicular to the radius vector and in the plane of motion) directions. The ratio α of the thrust acceleration to the gravity acceleration in the initial orbit is assumed to be small compared to one.

Practical interest in these two problems stems from the fact that the optimum steering program, to achieve escape from a circular orbit in minimum time, is closely approximated by the tangential and circumferential steering programs when α is small. Indeed, Lawden[8,9] has shown that the optimum instantaneous thrust direction lies between the tangential and circumferential programs.

The problem of analytically describing the motion with either tangential or circumferential thrust acceleration is also of considerable mathematical interest. It is known that the condition for escape from the point of central attraction occurs in a finite time. The difficulty in solving this problem stems from the singularity at the instant of escape. In general, it is necessary to treat one or more of the variables as being small in order to integrate the equations. The assumption usually breaks down in the vicinity of escape. A brief review of the past approaches is given in the following pages.



a) $\alpha<\alpha\ (0)$

b) $\alpha=\alpha(0)$

c) $\alpha\ (0)<\alpha<0$

d) $0<\alpha<\ (2)$

e) $\alpha\ =\ \alpha\ (2)$

f) $\alpha>\alpha\ (\ 2\ )$

FIGURE 6. TOTALITY OF MOTIONS

Tsien[10] in 1953 obtained a crude approximation for the circumferential case by neglecting the second derivative of the radial distance with respect to time. He found that the ratio ρ of the radial distance to its initial value is given by

$$\rho = (1 - \alpha\tau)^{-2} \quad ,$$

where τ is the time in units of the initial orbit period. In addition, the ratio z of the square of angular momentum to its initial value is equal to ρ in his analysis. Tsien also predicted the time to escape:

$$\tau_E = \frac{1 - (2\alpha)^{1/4}}{\alpha} \quad .$$

Benney[11] obtained in the tangential case a solution for ρ and z in powers of α with the path length s (normalized on the circumference of the initial orbit) as the independent parameter. To the first-order he obtained

$$\rho = 1 + 2\alpha \; (s - \sin s), \qquad z = 1 + 2\alpha s.$$

However, in Section 3 it is shown that to first-order $s = \tau$ in which case Benney's solutions are

$$\rho = 1 + 2\alpha(\tau - \sin\tau), \qquad z = 1 + 2\alpha\tau \; .$$

The escape time

$$\tau_E = \frac{1 - (20)^{1/8}\alpha^{1/4}}{\alpha}$$

predicted by Benney is only slightly smaller than τ_E predicted by Tsien for circumferential thrust.

Levin[4,12] obtained a solution for the circumferential case in powers of α. To first-order in α his results are identical to Benney's. These solutions are good approximations in the region $\alpha\tau << 1$. To extend this region closer to 1 higher order terms in α must be retained, which is a weakness in this method.

Billik[13] hoped to extend the region of good approximation farther from the original circular orbit in the circumferential case. He used z - 1 as a dependent variable and assumed it to remain small compared to 1 over the range of interest. However, one need only examine Tsien's solution to see that z - 1 can be even greater than 1 before escape is achieved.

Lass and Lorell[5] used the asymptotic method due to Kryloff and Bogoliuboff to obtain the differential equations for the first approximation in the circumferential case. If one assumes that the eccentricity of the instantaneous Kepler ellipse remains small, then the equations obtained by Lass and Lorell provide

$$\rho = z = (1 - 4\,\alpha\,\theta)^{-1/2},$$

where θ is the polar angle. The advantage of the asymptotic method, which was used here, over the classical perturbation method invoked by both Benney and Levin is that an infinite series in powers of α is not required to obtain accuracy to order α over the entire region (from circular orbit departure to escape).

Zee[14] investigated the problem of constant tangential thrust (not constant thrust acceleration). However, the orbit equation $\rho = \rho(\theta)$, $z = z(\theta)$ is the same for both cases; only the time along the orbit is different. He, too, used an asymptotic method to obtain the solutions

$$\frac{1}{\rho} = \frac{1}{z}\left(1 - \frac{2\,\alpha}{z^{1/2}}\sin\theta\right), \qquad z = (1 - 4\,\alpha\,\theta)^{-1/2}.$$

Observe that these results are the same as obtained by Lass and Lorell except for the decaying sinusoidal oscillation in the expression for x. However, no consideration was given to the higher harmonics in θ . Since these terms can conceivably contribute additional terms of order α , the preceding expressions for ρ and z might be incomplete to order α .

The following pages include the derivations of the complete first-order solution for ρ and z in terms of θ for both the tangential and circumferential cases. However, the difficulty of predicting the time of escape to order α has not been resolved, and further analysis is required in this area.

2.3.1 EQUATIONS OF MOTION

A particle P is moving in a circular Kepler orbit about the point of central attraction 0. At time t = 0 a constant thrust acceleration f is applied in either the tangential or circumferential direction. The equations of motion will be so formulated that when f = 0 they are linearized and integrable.

The position of P is defined with respect to 0 by the plane polar coordinates r and θ (Figure 7). For tangential thrust f lies along $\dot{s}$ and for circumferential thrust f lies along $r\dot{\theta}$. The unit of mass is selected to make the gravity constant unity. The instantaneous energy and angular momentum are then given by

$$E = \frac{1}{2}\dot{s}^2 - \frac{1}{r} , \qquad h = r^2\dot{\theta}$$

respectively. The time rates of change of E and h are

$$\text{tangential:} \qquad \dot{E} = f\dot{s}, \quad h = f\frac{h}{\dot{s}}$$

$$\text{circumferential:} \qquad \dot{E} = f\frac{h}{r}, \quad \dot{h} = fr$$

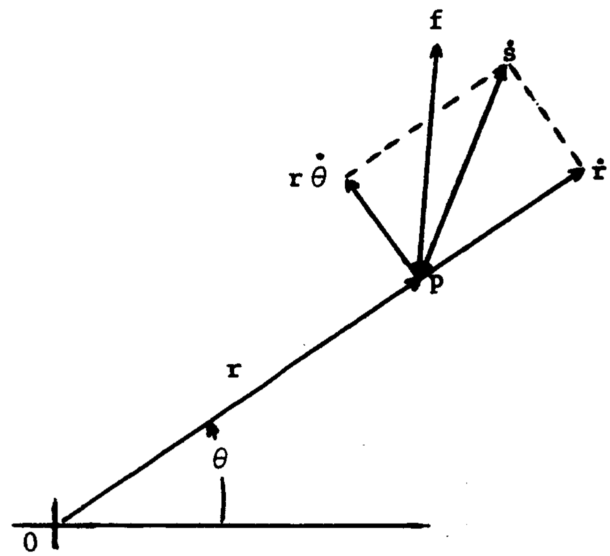


FIGURE 7. TANGENTIAL AND CIRCUMFERENTIAL THRUST GEOMETRY

To obtain the desired forms we replace r by u^{-1} and treat θ rather than t as the independent parameter. Using the operator equation

$$\frac{d}{dt}(\ ) = hu^2 \frac{d}{d\theta}(\ )$$

we obtain

$$\dot{r} = -hv, \qquad v = \frac{du}{d\theta}$$

$$\dot{s}^2 \equiv \dot{r}^2 + \frac{h^2}{r^2} = h^2(v^2 + u^2)$$

$$\dot{E} \equiv hu^2 \frac{dE}{d\theta}, \qquad E(u,v,h) = \frac{h^2}{2}(v^2 + u^2) - u.$$

Thus,

tangential: $$\frac{dE}{d\theta} = \frac{f\sqrt{v^2 + u^2}}{u^2}, \quad \frac{dh}{d\theta} = \frac{f}{hu^2\sqrt{v^2 + u^2}}$$

circumferential: $$\frac{dE}{d\theta} = \frac{f}{u}, \quad \frac{dh}{d\theta} = \frac{f}{hu^3}$$

The next step is to perform the differentiation

$$\frac{dE}{d\theta} = \frac{\partial E}{\partial u}\frac{du}{d\theta} + \frac{\partial E}{\partial v}\frac{dv}{d\theta} + \frac{\partial E}{\partial h}\frac{dh}{d\theta}$$

and obtain the orbit equation for the two cases. It is convenient at this point to introduce $w = h^2$. The complete set of equations is given below.

tangential: $$\frac{du}{d\theta} = v, \quad \frac{dv}{d\theta} + u = \frac{1}{w}$$

$$\frac{dw}{d\theta} = \frac{2f}{u^2\sqrt{v^2 + u^2}}, \quad \frac{dt}{d\theta} = \frac{1}{u^2\sqrt{w}}$$

circumferential: $$\frac{du}{d\theta} = v, \quad \frac{dv}{d\theta} + u = \frac{1}{w} - \frac{fv}{wu^3}$$

$$\frac{dw}{d\theta} = \frac{2f}{u^3}, \quad \frac{dt}{d\theta} = \frac{1}{u^2\sqrt{w}}$$

The initial conditions (at $\theta = 0$) for a circular orbit are

$$u(0) = u_o, \qquad v(0) = 0, \qquad w(0) = u_o^{-1}$$

The equations for u, v, and w are independent of t, and hence represent a complete third-order system. Dividing each equation by u_o and introducing the nondimensional parameters:

$$x = \frac{u}{u_o} , \qquad y = \frac{v}{u_o} , \qquad z = wu_o , \qquad \alpha = \frac{f}{u_o^2}$$

we have

tangential:

$$\frac{dx}{d\theta} = y , \quad \frac{dy}{d\theta} + x = \frac{1}{z} , \quad \frac{dz}{d\theta} = \frac{2\alpha}{x^2\sqrt{y^2 + x^2}} \tag{27}$$

circumferential:

$$\frac{dx}{d\theta} = y , \quad \frac{dy}{d\theta} + x = \frac{1}{z} - \frac{\alpha y}{zx^3} , \quad \frac{dz}{d\theta} = \frac{2\alpha}{x^3} \tag{28}$$

With $\tau = u_o^{3/2}$ t, the time equation can also be written in a nondimensional form

$$\frac{d\tau}{d\theta} = \frac{1}{x^2 \sqrt{z}}$$

Then if one can obtain the solutions $x = x(\theta)$ and $z = z(\theta)$, τ can be reduced to the quadrature

$$\tau = \int \frac{d\theta}{x^2(\theta)\sqrt{z(\theta)}} + \text{constant} \tag{29}$$

2.3.2 .FIRST APPROXIMATION

When $\alpha = 0$, equations (27) and (28) have solutions of the form

$$x = \frac{1}{\gamma} + \mu\cos(\theta - \beta) , \quad y = -\mu\sin(\theta - \beta), \quad z = \gamma \tag{30}$$

where μ, β and γ are constants of integration. The method of Kryloff and Bogoliuboff[15] will now be applied to obtain asymptotic solutions to equations (27) and (28) when $\alpha \neq 0$ but is sufficiently small. When $\alpha \neq 0$, regard equation (30) as a set of transformation equations to the new variables μ, β and γ.

(1) <u>Tangential Acceleration</u>

In terms of the new variables equation (27) becomes

$$\frac{d\mu}{d\theta} \cos \phi + \mu \frac{d\beta}{d\theta} \sin \phi = \frac{1}{\gamma^2} \frac{d\gamma}{d\theta}$$

$$\frac{d\mu}{d\theta} \sin\phi + \mu \frac{d\beta}{d\theta} \cos \phi = 0$$

$$\frac{d\gamma}{d\theta} = \frac{2 \alpha\gamma^3}{(1 + \mu\gamma \cos \phi)^2 \sqrt{1 + 2\mu\gamma \cos \phi + \mu^2 \gamma^2}}$$

where $\phi = \theta - \beta$. These equations are intractable in their present form. However, if the assumption is made that $\mu\gamma << 1$ over the region of interest, an asymptotic solution can be obtained. Later, a verification will be given that for $\alpha > 0$, the quantity $\mu\gamma$ decays from an initial amplitude of order α to zero as the condition for escape is approached. The equations can, therefore, be written as,

$$\left.\begin{aligned} \frac{d\mu}{d\theta} &= 2\alpha\gamma(1 - 3\mu\gamma\cos \phi) \cos \phi \\ \frac{d\beta}{d\theta} &= 2\alpha \frac{\gamma}{\mu} (1 - 3\mu\gamma\cos \phi) \sin \phi \\ \frac{d\gamma}{d\theta} &= 2 \alpha\gamma^3 (1 - 3\mu\gamma\cos \phi) \end{aligned}\right\} \quad (31)$$

The functions on the right-hand side of equation (29) are next represented by a trigonometric series with

$$\left.\begin{aligned}
f_1(\phi) &= (1-3\,\mu\gamma\cos\phi)\cos\phi = b_o^{(1)} + \sum_{n=1}^{\infty}(b_n^{(1)}\cos n\,\phi + a_n^{(1)}\sin n\,\phi) \\
f_2(\phi) &= (1-3\,\mu\gamma\cos\phi)\sin\phi = b_o^{(2)} + \sum_{n=1}^{\infty}(b_n^{(2)}\cos n\,\phi + a_n^{(2)}\sin n\,\phi) \\
f_3(\phi) &= (1-3\,\mu\gamma\cos\phi) \qquad\quad = b_o^{(3)} + \sum_{n=1}^{\infty}(b_n^{(3)}\cos n\,\phi + a_n^{(3)}\sin n\,\phi)
\end{aligned}\right\} \quad (32)$$

where

$$b_o^{(i)} = \frac{1}{2\pi}\int_o^{2\pi} f_i(\phi)\,d\phi\ , \qquad b_n^{(i)} = \frac{1}{\pi}\int_o^{2\pi} f_i(\phi)\cos n\phi\,d\phi$$

$$a_n^{(i)} = \frac{1}{\pi}\int_o^{2\pi} f_i(\phi)\sin n\,\phi\,d\phi \qquad i = 1,2,3 \qquad (33)$$

Since α is small, μ, β and γ will be slowly changing functions of θ. The first approximations are obtained by setting

$$\mu = \overline{\mu}\ , \qquad \beta = \overline{\beta}\ , \qquad \gamma = \overline{\gamma} \qquad (34)$$

where $\overline{\mu}$, $\overline{\beta}$ and $\overline{\gamma}$ are determined by neglecting the higher harmonics in equations (32), retaining only $b_o^{(i)}$ and holding them $(\overline{\mu}, \overline{\beta}, \overline{\gamma})$ constant over the interval $\phi = 0$ to 2π. The differential equations for the first approximation are

$$\left.\begin{aligned}
\frac{d\overline{\mu}}{d\theta} &= \frac{\alpha\overline{\gamma}}{\pi}\int_o^{2\pi}(1 - 3\,\overline{\mu\gamma}\cos\phi)\cos\phi\,d\phi = -3\,\alpha\overline{\mu\gamma}^2 \\
\frac{d\overline{\beta}}{d\theta} &+ \frac{\alpha\overline{\gamma}}{\overline{\mu}\pi}\int_o^{2\pi}(1 - 3\,\overline{\mu\gamma}\cos\phi)\sin\phi\,d\phi = 0 \\
\frac{d\overline{\gamma}}{d\theta} &= \frac{\alpha\overline{\gamma}^3}{\pi}\int_o^{2\pi}(1 - 3\overline{\mu\gamma}\cos\phi)\,d\phi = 2\,\alpha\overline{\gamma}^3
\end{aligned}\right\} \quad (35)$$

where $\phi = \theta - \overline{\beta}$. The last equation has the solution

$$\overline{\gamma} = (1 - 4\alpha\theta)^{-1/2} \tag{36}$$

Combining the first and last equations in (35) and integrating one gets

$$\overline{\mu} = K\overline{\gamma}^{-3/2} \tag{37}$$

where K is the constant of integration.

(2) Circumferential Acceleration

Solutions similar to equations (36) and (37) can be obtained for the circumferential case. The differential equations

$$\left.\begin{aligned}
\frac{d\mu}{d\theta} &= 2\alpha\gamma\left[(1 - 3\mu\gamma\cos\phi)\cos\phi - \frac{\mu\gamma}{2}\sin^2\phi\right] = 2\alpha\gamma\, g_1(\phi)\\
\frac{d\beta}{d\theta} &= \frac{2\alpha\gamma}{\alpha}\quad(1 - 3\mu\gamma\cos\phi)\sin\phi + \frac{\mu\gamma}{2}\sin\phi\cos\phi = \frac{2\alpha\gamma}{\mu}\, g_2(\phi)\\
\frac{d\gamma}{d\theta} &= 2\alpha\gamma^3(1 - 3\mu\gamma\cos\phi)\ 2\alpha\gamma^3 f_3(\phi)
\end{aligned}\right\} \tag{38}$$

are obtained by substituting equation (30) into equation (28) and by dropping terms of order $(\alpha\gamma)^2$. The third equation in (38) is the same as in the tangential case.

As before, the right-hand sides are represented by trigonometric series of the form in equation (31) except that $g_1(\phi)$ replaces $f_1(\phi)$ and $g_2(\phi)$ replaces $f_2(\phi)$. For the first approximation

$$\left.\begin{aligned}
\frac{d\overline{\mu}}{d\theta} &= \frac{\alpha\overline{\gamma}}{\pi}\int_0^{2\pi}\left[(1 - 3\,\overline{\mu\gamma}\cos\phi)\cos\phi - \frac{\overline{\mu\gamma}}{2}\sin^2\phi\right]d\phi = \frac{7}{2}\alpha\overline{\mu\gamma}\\
\frac{d\overline{\beta}}{d\theta} &= \frac{\alpha\overline{\gamma}}{\mu\pi}\int_0^{2\pi}\left[(1 - 3\,\overline{\mu\gamma}\cos\phi)\sin\phi + \frac{\overline{\mu\gamma}}{2}\sin\phi\cos\phi\right]d\phi = 0\\
\frac{d\overline{\gamma}}{d\theta} &= \frac{\alpha\overline{\gamma}^3}{\pi}\int_0^{2\pi}(1 - 3\,\mu\gamma\cos\phi)\,d\phi = 2f\overline{\gamma}^3
\end{aligned}\right\} \tag{39}$$

from which

$$\overline{\gamma} = (1 - 4\alpha\theta)^{-1/2} \tag{40}$$

$$\overline{\mu} = K\overline{\gamma}^{-7/4} \tag{41}$$

2.3.3 IMPROVED FIRST APPROXIMATION

In obtaining $\overline{\mu}$, $\overline{\beta}$ and $\overline{\gamma}$ as functions of θ the higher harmonics were ignored in the series equations (32). The first-order effect of these terms may be determined from the improved solutions (improved over equations (34))

$$\mu = \overline{\mu} + 2\alpha\overline{\gamma}\sum_{n=1}^{\infty}\frac{1}{n}\left(b_n^{(1)}\sin n\phi - a_n^{(1)}\cos n\phi\right)$$

$$\beta = \overline{\beta} + \frac{2\alpha\overline{\gamma}}{\mu}\sum_{n=1}^{\infty}\frac{1}{n}\left(b_n^{(2)}\sin n\phi - a_n^{(2)}\cos n\phi\right) \tag{42}$$

$$\gamma = \overline{\gamma} + 2\alpha\overline{\gamma}^3\sum_{n=1}^{\infty}\frac{1}{n}\left(b_n^{(3)}\sin n\phi - a_n^{(3)}\cos n\phi\right)$$

where setting $\mu = \overline{\mu}$, $\beta = \overline{\beta}$ and $\gamma = \overline{\gamma}$ in the coefficients $b_n^{(i)}$ and $a_n^{(i)}$.

(1) Tangential Acceleration

In this case

$$b_n^{(1)} = \frac{1}{\pi}\int_0^{2\pi}(1 - 3\,\overline{\mu\gamma}\cos\phi)\cos\phi\cos n\phi\,d\phi$$

$$b_1^{(1)} = 1\ , \qquad b_n^{(1)} = -\frac{3\overline{\mu\gamma}}{2}\ , \qquad \text{others} = 0$$

$$a_n^{(1)} = \frac{1}{\pi}\int_0^{2\pi}(1 - 3\,\overline{\mu\gamma}\cos\phi)\cos\phi\sin n\phi\,d\phi = 0$$

$$b_n^{(2)} = \frac{1}{\pi}\int_0^{2\pi}(1 - 3\,\overline{\mu\gamma}\cos\phi)\sin\phi\cos n\phi\,d\phi = 0$$

$$a_n^{(2)} = \frac{1}{\pi}\int_0^{2\pi}(1 - 3\,\overline{\mu\gamma}\cos\phi)\sin\phi\sin n\phi\,d\phi$$

$$a_1^{(2)} = 1 \quad , \quad a_2^{(2)} = - \frac{3\,\overline{\mu\gamma}}{2} \quad , \quad \text{others} = 0$$

$$b_n^{(3)} = \frac{1}{\pi} \int_o^{2\pi} (1 - 3\,\overline{\mu\gamma} \cos \phi) \cos n\phi \, d\phi$$

$$b_1^{(3)} = - 3\,\overline{\mu\gamma} \quad , \quad \text{others} = 0$$

$$a_n^{(3)} = \frac{1}{\pi} \int_o^{2\pi} (1 - 3\,\overline{\mu\gamma} \cos \phi) \sin n\phi \, d\phi = 0$$

and equations(42) become

$$\left.\begin{aligned} \mu &= \overline{\mu} + 2\,\alpha\overline{\gamma}\,(\sin \phi - \frac{3\,\overline{\mu\gamma}}{4} \sin 2\phi) \\ \beta &= \overline{\beta} + 2\,\alpha\,\frac{\overline{\gamma}}{\overline{\alpha}}\,(- \cos \phi + \frac{3\,\overline{\mu\gamma}}{4} \cos 2\phi) \\ \gamma &= \overline{\gamma}\,(1 - 6\,\alpha\overline{\mu\gamma}^{\,3} \sin\phi) \end{aligned}\right\} \qquad (43)$$

When equations (43) and (37) are substituted into the equation for x in (30), and when terms of order α^2 are dropped, the equation for x becomes

$$x = \frac{1}{\overline{\gamma}} \left[1 + \frac{K}{\overline{\gamma}^{1/2}} \cos\,(\theta - \overline{\beta}) \right] \qquad (44)$$

Using the relation

$$\frac{d\overline{\gamma}}{d\theta} = 2\,\alpha\overline{\gamma}^3 \quad ,$$

then the derivative of equation (44) with respect to θ is

$$\frac{dx}{d\theta} = -2\,\mu\bar{\gamma} - 6\mu K\,\bar{\gamma}^{1/2}\cos(\theta - \bar{\beta}) - \frac{K}{\bar{\gamma}^{3/2}}\sin(\theta - \bar{\beta}).$$

The constants K and $\bar{\beta}$ are determined from the initial conditions

$$x = 1 \quad , \frac{dx}{d\theta} = 0 \quad , \quad \bar{\gamma} = 1 \qquad \text{at } \theta = 0$$

Finding $K = 2\alpha$ and β in $\pi/2$, equation (44) becomes

$$x = \frac{1}{\bar{\gamma}}\left(1 + \frac{2}{\bar{\gamma}^{1/2}}\sin\theta\right) \tag{45}$$

The final results are then

$$x = \frac{1}{z}\left(1 + \frac{2\alpha}{z^{1/2}}\sin\theta\right) \quad , \quad z = (1 - 4\alpha\theta)^{-1/2} \tag{46}$$

which is the same as obtained by Zee[14]. However, the present analysis has confirmed that the higher harmonics do generate first order terms in α but that these terms cancel.

This analysis was based on the assumption that $\mu\gamma \ll 1$. The correctness may be seen from

$$\overline{\mu\gamma} = 2\,\alpha(1 - 4\,\alpha\,\theta)^{1/4} \tag{47}$$

which is $\ll 1$ for $\alpha \ll 1$ and which tends to zero as θ approaches $\frac{1}{4\alpha}$ for positive values of α.

(2) Circumferential Acceleration

In place of equations in (43), the improved solutions for the circumferential case become:

$$\left.\begin{aligned}
\mu &= \overline{\mu} + 2\,\alpha\,\overline{\gamma}\,(\sin\phi - \frac{5\,\overline{\mu\gamma}}{8}\,\sin 2\phi) \\
\beta &= \overline{\beta} + \frac{2\,\alpha\,\overline{\gamma}}{\overline{\mu}}(-\cos\phi + \frac{5\,\overline{\mu\gamma}}{8}\,\cos 2\phi) \\
\gamma &= \overline{\gamma}\,(1 - 6\,\alpha\overline{\mu\gamma}^{\,3}\,\sin\phi)
\end{aligned}\right\} \qquad (48)$$

where $\phi = \theta - \overline{\beta}$. The resulting first-order solution is found to be

$$x = \frac{1}{z}\,(1 + \frac{2\alpha}{z^{3/4}}\,\sin\theta)\quad , \qquad z = (1 - 4\alpha\,\theta\,)^{-1/2} \qquad (49)$$

This solution differs from equation in (46) for the tangential case only in the decaying sinusoidal oscillation of order α in x. Hence, the difference in tangential and circumferential thrust is small for small α.

In this case the stipulation that $\mu\gamma << 1$ is confirmed by

$$\overline{\mu\gamma} = 2\,\alpha\,(1 - 4\,\alpha\theta\,)^{3/8} \qquad (50)$$

2.3.4 TIME EQUATION

Having determined x and z as functions of θ , for the tangential and circumferential cases, it is possible to integrate the time expression, equation (29), by elementary functions. For convenience, equation (29) is rewritten as

$$\tau = \int \frac{d\theta}{x^2(\theta)\sqrt{z(\theta)}} + \text{ constant} \; . \qquad (51)$$

When the time expression is evaluated for the tangential case, using the equations in (46), the resulting integral becomes

$$\tau = \int (1-4\alpha\theta)^{-3/4} d\theta - 4\alpha \int \sin\theta\,(1-4\alpha\,\theta)^{-1/2} d\theta + 12\,\alpha^2 \int \sin^2\theta (1-4\alpha\theta)^{-1/4} d\theta$$

$$+ \text{ constant} \; .$$

Upon integrating this expression and evaluating the constant of integration with the initial conditions $\tau = 0$, $\theta = 0$, the complete first-order solution becomes

$$\tau = \frac{1}{\alpha} \left[1-(1-4\alpha\theta)^{1/4} \right] -4\alpha(1-\cos\theta) + 0(\alpha^2) . \tag{52}$$

When the time expression is evaluated for the circumferential case, using the equation in (49), the resulting integral becomes

$$\tau = \int (1-4\alpha\theta)^{-3/4} d\theta - 4\alpha \int \sin\theta (1-4\alpha\theta)^{-3/8} d\theta + 12\alpha^2 \int \sin^2\theta \, d\theta$$

$$+ \text{constant}.$$

Upon integrating this expression and evaluating the constant of integration with the initial conditions $\tau = 0$, $\theta = 0$, the complete first-order solution becomes

$$\tau = \frac{1}{\alpha} \left[1-(1-4\alpha\theta)^{1/4}\right] -4\alpha(1-\cos\theta) + 0(\alpha^2) . \tag{53}$$

Equations (52) and (53) show that, to the first order, the time expressions for the tangential and circumferential cases are the same. If a second-order theory were carried out, the second-order part of equations (52) and (53) would be different.

SECTION 3

PERTURBATION SOLUTIONS OF THE EQUATIONS OF MOTION

The intent of this investigation has been to improve the analytical representation of low-thrust trajectories through perturbation solutions to the system equations of motion. Of primary interest is the application of the perturbation solutions to orbit transfer problems. By making use of these solutions, certain optimization problems of interest may be treated within the realm of simple optimization theory and improved numerical computation techniques can be developed.

3.1 PERTURBATION SOLUTIONS STARTING FROM A CIRCULAR ORBIT

This subsection summarizes the perturbation theory, which includes a complete second-order theory for the motion of a vehicle where the thrust vector is maintained at a constant angle with respect to the radius vector. The following pages will include the system equations, a first-order solution for tangential thrusting in order to illustrate the basic methodology, and then a second-order theory development for thrust in an arbitrary direction. A portion of this subsection will be devoted to discussing the numerical results for representative low-thrust trajectories, and also the application of the theory to orbit transfer problems using an energy/momentum approach.

3.1.1 EQUATIONS OF MOTION

Consider the problem of finding perturbation solutions of the differential equations of motion of a vehicle moving under low thrust. The equations of motion are

$$\frac{d^2\rho}{d\tau^2} - \frac{\nu^2}{\rho} + \frac{1}{\rho^2} = \alpha \cos \psi \tag{54}$$

$$\frac{1}{\rho}\frac{d}{d\tau}(\rho\nu) = \alpha \sin \psi . \tag{55}$$

The notation is that of E. Levin[4] where $\nu = \rho \frac{d\theta}{d\tau}$, $0 < \alpha << 1$, and ρ, θ, τ are dimensionless position and time variables (see Figure 8).

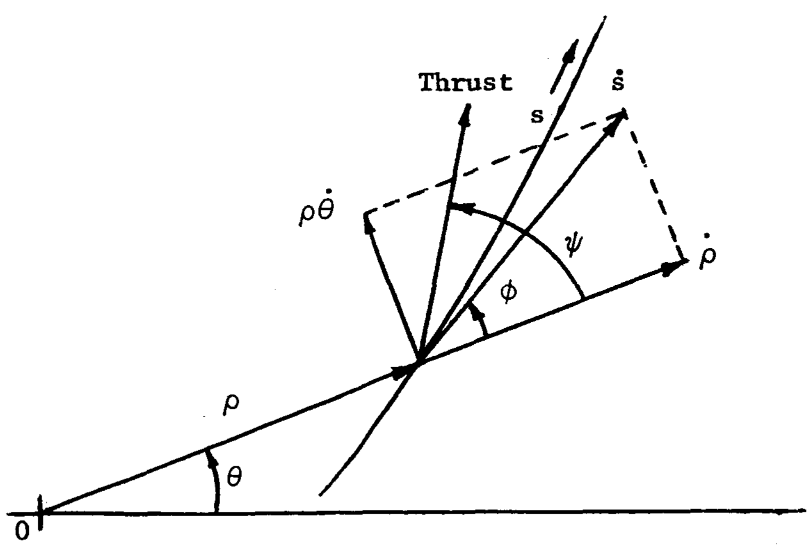


FIGURE 8. GENERAL LOW-THRUST GEOMETRY

The constants of integration are determined according to the initial conditions, $\rho = 1$, $\dot{\rho} = 0$, $\tau = 0$, $\dot{\theta} = 1$, $\theta = 0$, and $s = 0$, for the departure from a circular orbit.

3.1.2 THRUST IN TANGENTIAL DIRECTION

Consider the case of tangential acceleration: $\tan \psi = \frac{\nu}{\dot{\rho}}$. Then the equations of motion, (54) and (55), become

$$\frac{d^2\rho}{d\tau^2} - \frac{\nu^2}{\rho} + \frac{1}{\rho^2} = \frac{\alpha\dot{\rho}}{\sqrt{\dot{\rho}^2 + \nu^2}} \tag{56}$$

$$\frac{1}{\rho}\frac{d}{d\tau}(\rho\nu) = \frac{\alpha\nu}{\sqrt{\dot{\rho}^2 + \nu^2}} . \tag{57}$$

With $(ds)^2 = (d\rho)^2 + \rho^2(d\theta)^2$, the dimensionless arc length, and $\nu = \rho\frac{d\theta}{d\tau}$, the velocity becomes

$$\left(\frac{ds}{d\tau}\right)^2 = \left(\frac{d\rho}{d\tau}\right)^2 + \nu^2 = V^2 . \tag{58}$$

From Levin[4], the rate of change of the instantaneous energy E is

$$\dot{E} = \alpha\dot{\rho}\cos\psi + \alpha\nu\sin\psi . \tag{59}$$

For the tangential case, equation (59) becomes

$$\dot{E} = \alpha\dot{\rho}\left(\frac{\dot{\rho}}{V}\right) + \alpha\nu\left(\frac{\nu}{V}\right) = \alpha V . \tag{60}$$

Now, changing to the independent variable, s, we obtain

$$V\frac{dE}{ds} = \alpha V$$

$$E = \alpha s + E_o \; . \tag{61}$$

Expressing equation (61) in terms of the speed V for an initially circular orbit, the first integral obtained by Benney[11] is

$$V^2 = \left(\frac{ds}{d\tau}\right)^2 = \frac{2}{\rho} + 2\alpha s - 1 , \tag{62}$$

Thus equation (57) can be written as

$$\frac{d(\rho\nu)}{\rho\nu} = \frac{\alpha\, d\tau}{\sqrt{\dot{\rho}^2 + \nu^2}} = \frac{\alpha\, d\tau}{\frac{ds}{d\tau}} = \frac{\alpha\, ds}{\left(\frac{ds}{d\tau}\right)^2} = \frac{\alpha\, ds}{\frac{2}{\rho} + 2\alpha s - 1} \; . \tag{63}$$

Integrating equation (63), the angular momentum becomes

$$h = \rho\nu = \exp\left\{ \alpha \int \frac{ds}{\frac{2}{\rho} + 2\alpha s - 1} \right\} . \tag{64}$$

With the approximation $e^x = 1 + x + \frac{x^2}{2}$, for small x, equation (64) reduces to

$$h = \rho\nu = 1 + \alpha\, g(s) + \frac{\alpha^2}{2}\, g^2(s) \quad , \tag{65}$$

where $g(s) = \int \frac{ds}{\frac{2}{\rho} + 2\alpha s - 1}$.

Similarly, equation (56) can be written as

$$\frac{d^2\rho}{d\tau^2} - \frac{\nu^2}{\rho} + \frac{1}{\rho^2} = \alpha\, \frac{d\rho}{ds} \; . \tag{66}$$

The following will show that equations (56) and (57) can be brought into the form

$$\frac{d^2u}{d\theta^2} + u = \exp\left\{-2\alpha\int \frac{ds}{2u+2\alpha s-1}\right\} \tag{67}$$

$$\frac{d\tau}{d\theta} = \frac{1}{u^2}\exp\left\{-\alpha\int \frac{ds}{2u+2\alpha s-1}\right\} \tag{68}$$

where $u = \frac{1}{\rho}$. Setting $\rho = \frac{1}{u}$ in equation (64) yields

$$\rho\nu = \rho^2 \quad = \rho^2\frac{d\theta}{d\tau} = \frac{\dot{\theta}}{u^2} = e^{\alpha g(s)} = \frac{1}{u^2}\frac{d\theta}{d\tau} = h \tag{69}$$

which is equation (68). Setting $\rho = \frac{1}{u}$ in equation (56) yields

$$\frac{d\rho}{d\tau} = -\frac{1}{u^2}\frac{du}{d\theta}\frac{d\theta}{d\tau} = -h\frac{du}{d\theta}, \quad \frac{d^2\rho}{d\tau^2} = -h^2u^2\frac{d^2u}{d\theta^2} - hu^2\frac{dh}{d\theta}\frac{du}{d\theta}.$$

Substituting these derivatives into equation (66) the result is

$$\frac{d^2u}{d\theta^2} + u = \frac{1}{h^2} - \frac{1}{h}\left[\frac{dh}{d\theta}\frac{du}{d\theta} + \frac{\alpha}{hu^2}\frac{d\rho}{ds}\right]. \tag{70}$$

For the tangential case, the following analysis will verify that the quantity inside the bracket is zero. From equations (68), (69), and (62), it follows that

$$\frac{dh}{d\tau} = \frac{\alpha h}{\dot{s}^2}\frac{ds}{d\tau} = \frac{\alpha h}{\dot{s}} = \frac{dh}{d\theta}\dot{\theta} = hu^2\frac{dh}{d\theta} \quad \text{or} \quad \frac{dh}{d\theta} = \frac{\alpha}{u^2\dot{s}}.$$

Also $d\rho/ds$ can be written as $\frac{d\rho}{ds} = - \frac{1}{u_2} \frac{du}{d\theta} \frac{d\theta}{ds}$.

From equation (57) $\frac{d\theta}{ds} = \frac{u^2}{\alpha} \frac{dh}{d\tau} = \frac{h\,u^2}{\dot{s}}$

so that $\frac{d\rho}{ds} = - \frac{h}{\dot{s}} \frac{du}{d\theta}$.

Thus the quantity inside the bracket in equation (70) is zero, substantiating equation (67). Regarding the right side of equation (67) as a function of θ, the equation is now a nonhomogeneous linear differential equation with constant coefficients. The complete solution is obtainable by variation of parameters in the form

$$u = A \cos \theta + B \sin \theta - \frac{i}{2} e^{i\theta} \exp\left\{i\,\theta - 2\alpha\, g(s)\right\} d\theta + \frac{i}{2} e^{i\theta} \int \exp\left\{i\,\theta - 2\alpha\, g(s)\right\} \quad (71)$$

where $i = \sqrt{-1}$, A and B are arbitrary constants and $g(s) = \int \frac{ds}{2u+2\alpha\, s-1}$
Integrating equation (71) by parts, the result is

$$u = A \cos \theta + B \sin \theta + \exp \left\{-2\alpha\, g(s)\right\} + 0(\alpha^2) \quad (72)$$

for a first-order solution. To obtain a more explicit form for the particular integral, differentiate

$$u = \exp \left\{-2\alpha\, g(s)\right\} = \exp \left\{-2\alpha \int \frac{ds}{2u-2\,\alpha s-1}\right\} \quad (73)$$

with respect to s to obtain

$$\frac{du}{ds} = \frac{-2\alpha}{2u+2\alpha s-1} \exp\left\{-2\alpha\int\frac{ds}{2u+2\alpha s-1}\right\} = \frac{-2\alpha u}{2u+2\alpha s-1} \tag{74}$$

which can be arranged as

$$u\ ds + s\ du = \frac{1}{2\alpha}du - \frac{1}{\alpha}u\ du = d(us) \tag{75}$$

which has the solution $(us) = \frac{u}{2\alpha} - \frac{u^2}{2\alpha} + c_1$, or since $u = 1$ when $s = 0$,

$$u = 1-2\alpha s. \tag{76}$$

Using equation (76) in equation (68) and knowing that

$$d\theta = \sqrt{u^2 - \frac{1}{u^2}\left(\frac{du}{ds}\right)^2}\ ds\ , \quad \text{the result is} \quad \theta = s + 0(\alpha\ ),$$

and evaluating the constants of integration in equation (72) according to the initial conditions, the constants are $A = 0$ and $B = 2\alpha$, so that equation (72) becomes

$$u = 1 - 2\alpha(s - \sin s) + 0(\alpha^2)\ . \tag{77}$$

Since $\rho = \frac{1}{u}$, equation (77) is in agreement with Benney's result,

$$\rho = 1 + 2\alpha(s - \sin s) + 0(\alpha^2)\ . \tag{78}$$

Now writing equation (68) and neglecting terms of order α^2,

$$d\tau = [1 - 2\alpha(s - \sin s)]^{-2}(1 - \alpha s)\ d\theta + 0(\alpha^2) \tag{79}$$

$$= [1 + \alpha(3s - 4\sin s)]\ d\theta + 0(\alpha^2)\ .$$

Using $d\theta = \sqrt{u^2 - \frac{1}{u^2}\left(\frac{du}{ds}\right)^2}\ ds$, we get

$$d\theta = [1 - 2\alpha(s - \sin s)]\ ds + 0(\alpha^2) \tag{80}$$

Equation (79) can now be integrated to obtain

$$\tau\text{-}C = \int [1 + \alpha(3s - 4\sin s)]\ [1 - 2\alpha(s - \sin s)]\ ds + 0(\alpha^2)$$

$$= s + \alpha\left(\frac{s^2}{2} + 2\cos s\right) + 0(\alpha^2) \ . \tag{81}$$

When $\tau = 0$, $s = 0$ so that $C = -2\alpha$ and equation (81) can be written as

$$\tau = s + \alpha\left(\frac{s^2}{2} - 4\sin^2\frac{s}{2}\right) + 0(\alpha^2) \ . \tag{82}$$

Equations (78) and (82) constitute a complete first-order solution of equations (67) and (68) in the case of tangential acceleration. Analogously, a second-order solution can be derived.

3.1.3 THRUST IN ARBITRARY DIRECTION

Now consider the more general case of thrusting with a constant orientation angle ψ. The equations of motion are of the form of equations (54) and (55). Writing equation (59) in a different form, we have

$$\dot{E} = V\frac{dE}{ds} = \alpha\cos(\phi - \psi)\sqrt{\dot{\rho}^2 + \nu^2} \tag{83}$$

or

$$E = \alpha\int\cos(\phi - \psi)\ ds + C = \frac{1}{2}V^2 - \frac{1}{\rho} = \frac{1}{2}\dot{s}^2 - \frac{1}{\rho} \ . \tag{84}$$

Choosing $C = -\frac{1}{2}$ to satisfy initial conditions, $\rho = 1$, $s = 0, \tau = 0$, then equation (84) becomes the first integral

$$V^2 = \dot{s}^2 = \frac{2}{\rho} + 2\alpha f(s) - 1 \tag{85}$$

where $f(s) = \int \cos(\phi - \psi)\, ds = s \sin\psi + 0(\alpha)$.

In the same way that equation (64) was obtained, the angular momentum becomes

$$h = \rho\nu = \exp\left\{ \alpha \int \frac{\sin\psi}{\nu\, V}\, ds \right\} . \tag{86}$$

Equation (70) will now become

$$\frac{d^2u}{d\theta^2} + u = \frac{1}{h^2} - \frac{1}{h}\left[\frac{dh}{d\theta}\frac{du}{d\theta} + \frac{\alpha\cos\psi}{h\, u^2}\right] . \tag{87}$$

Since $h = \frac{1}{u^2}\frac{d\theta}{d\tau}$ and $\frac{dh}{d\theta} = \frac{\alpha \sin\psi}{uV} \sqrt{\left(\frac{d\rho}{d\theta}\right)^2 + \rho^2}$,

equation (87) becomes

$$\frac{d^2u}{d\theta^2} + u = \frac{1}{h^2} - \frac{\alpha}{hV}\left[\frac{\sin\psi}{u}\frac{du}{d\theta}\sqrt{\left(\frac{d\rho}{d\theta}\right)^2 + \rho^2} + \cos\frac{ds}{d\theta}\right]. \tag{88}$$

Hence, equations (88) and (86) can be written as

$$\frac{d^2u}{d\theta^2} + u = \exp\left\{-2\alpha\int\frac{\sin\psi\, ds}{\nu\, V}\right\} - \frac{\alpha}{V}\left[\frac{1}{u}\sin\psi\sqrt{\left(\frac{d\rho}{d\theta}\right)^2 + \rho^2}\,\frac{du}{d\theta} + \cos\psi\,\frac{ds}{d\theta}\right] \exp\left\{-\alpha\int\frac{\sin\psi\, ds}{\nu\, V}\right\} , \tag{89}$$

$$\frac{d\tau}{d\theta} = \frac{1}{u^2} \exp\left\{ -\alpha \int \frac{\sin\psi\, ds}{\nu V} \right\} . \qquad (90)$$

The complete solution of equation (89) is obtainable by variation of parameters, analogously to equation (71), as

$$u = A \cos\theta + B \sin\theta - \frac{i}{2} e^{i\theta} \int e^{-i\theta} F_1(\theta)\, d\theta + \frac{i}{2} e^{-i\theta} \int e^{i\theta} F_1(\theta)\, d\theta \qquad (91)$$

where

$$F_1(\theta) = \exp\left\{ -2\int \frac{\alpha \sin\psi}{\nu V} \frac{ds}{d\theta}\, d\theta \right\} - \frac{\alpha}{V} \left[\frac{1}{u} \sin\psi \sqrt{\left(\frac{d\beta}{d\theta}\right)^2 + \rho^2}\, \frac{du}{d\theta} + \cos\psi \frac{ds}{d\theta} \right] \exp\left\{ -\int \frac{\alpha \sin\psi}{\nu V} \frac{ds}{d\theta}\, d\theta \right\}$$

Using the approximation $e^x \approx 1 + x + \frac{x^2}{2}$ for small x, a first-order expression for equation (91) becomes

$$u = A \cos\theta + B \sin\theta - \frac{i}{2} e^{i\theta} \int e^{-i\theta} F_2(\theta)\, d\theta + \frac{i}{2} e^{-i\theta} \int e^{i\theta} F_2(\theta)\, d\theta \qquad (92)$$

where $F_2(\theta) = 1 - 2\alpha \int \frac{\sin\psi}{\nu V} \frac{ds}{d\theta}\, d\theta - \frac{\alpha}{V} \cos\psi \frac{ds}{d\theta} + 0(\alpha^2)$.

Upon integrating equation (92) and evaluating the constants of integration, A and B, according to the initial conditions $\theta = 0$, $u = 1$, $\frac{du}{d\theta} = 0$, and letting $\psi = \psi_o$, a constant, the result is

$$u = 1 - \alpha \left[2 \sin\psi_o (\theta - \sin\theta) + \cos\psi_o (1 - \cos\theta) \right] + 0(\alpha^2) \qquad (93)$$

$$\rho = 1 + \alpha[2 \sin\psi_o (\theta - \sin\theta) + \cos\psi_o (1 - \cos\theta)] + 0(\alpha^2). \quad (94)$$

Using equation (90) it follows that

$$\tau = \theta + \alpha[\sin\psi_o(\tfrac{3}{2}\theta^2 + 4\cos\theta - 4) + 2\cos\psi_o (\theta - \sin\theta)] + 0(\alpha^2). (95)$$

Since $(ds)^2 = (d\rho)^2 + \rho^2 (d\theta)^2$, then $ds = \rho d\theta + 0(\alpha^2)$ so that by equation (94)

$$s = \theta + \alpha[\sin\psi_o (\theta^2 + 2\cos\theta - 2) + \cos\psi_o (\theta - \sin\theta)] + 0(\alpha^2) . \quad (96)$$

In general, the basic solution to equations (54) and (55) can be written in any of the two explicit forms.

$$\rho = \rho(\theta), \tau = \tau(\theta) \quad (97)$$

$$\rho = \rho(\tau), \theta = \theta(\tau) \quad (98)$$

In the previous pages it has been shown that equations (94) and (95) take the form of (97). From equations (94) and (95) the form of (98) is obtained, namely,

$$\rho = 1 + \alpha[2 \sin\psi_o (\tau - \sin\tau) + \cos\psi_o (1 - \cos\tau) + 0(\alpha^2) (99)$$

and

$$\theta = \tau - \alpha[\sin\psi_o(\tfrac{3}{2}\tau^2 + 4\cos\tau - 4) + 2\cos\psi_o (\tau - \sin\tau)] + 0(\alpha^2). (100)$$

Other relations involving s are

$$\theta = s - \alpha[\sin \psi_o(s^2 + 2 \cos s - 2) + \cos \psi_o(s - \sin s)] + 0(\alpha^2), \qquad (101)$$

$$\rho = 1 + \alpha[2 \sin \psi_o(s - \sin s) + \cos \psi_o(1 - \cos s)] + 0(\alpha^2), \qquad (102)$$

$$\tau = s + \alpha[\sin \psi_o (\tfrac{1}{2} s^2 + 2 \cos s - 2) + \cos \psi_o(s - \sin s)] + 0(\alpha^2). \qquad (103)$$

The development of second-order expressions in the two explicit forms as noted by (97) and (98) will now be obtained. From earlier first-order results the following expressions are obtained:

$$u = 1 - \alpha[2 \sin \psi_o(\theta - \sin \theta) + \cos \psi_o(1 - \cos \theta)] + 0(\alpha^2)$$

$$\frac{du}{d\theta} = \alpha [- 2 \sin \psi_o(1 - \cos \theta) - \cos \psi_o \sin \theta] + 0(\alpha^2)$$

$$\rho = 1 + \alpha[2 \sin \psi_o(\theta - \sin \theta) + \cos \psi_o(1 - \cos \theta)] + 0(\alpha^2)$$

$$= \frac{ds}{d\theta} + 0(\alpha^2)$$

$$\frac{d\tau}{d\theta} = 1 + \alpha[\sin \psi_o(3\theta - 4 \sin \theta) + 2 \cos \psi_o(1 - \cos \theta)] + 0(\alpha^2)$$

$$\nu = 1 - \alpha[\sin \psi_o(\theta - 2 \sin \theta) + \cos \psi_o(1 - \cos \theta)] + 0(\alpha^2)$$

$$= V + 0(\alpha^2).$$

Upon substituting these expressions into equation (91) and integrating and evaluating the constants of integration, A and B, according to the initial conditions, the following is obtained:

$$u = 1 + \alpha[\cos \psi_o(\cos \theta - 1) - 2 \sin \psi_o(\theta - \sin \theta)] \qquad (104)$$

$$+ \alpha^2 [\sin^2 \psi_o(- 2 \theta^2 - 7 \theta \sin \theta - 18 \cos \theta + 18)$$

$$+ \cos^2 \psi_o(\theta \sin \theta + 2 \cos \theta - 2)$$

$$+ \sin \psi_o \cos \psi_o(- 8\theta - \tfrac{11}{2}\theta \cos \theta + \tfrac{27}{2} \sin \theta)] + 0(\alpha^3)$$

Since $(1+\alpha x+\alpha^2 y)^{-1} = 1-\alpha x+\alpha^2(x^2-y)+0(\alpha^3)$ equation (104) can be written as

$$\rho = 1+\alpha[\cos\psi_o(1-\cos\theta)+2\sin\psi_o(\theta-\sin\theta)] \tag{105}$$

$$+\alpha^2[\sin^2\psi_o(6\theta^2+4\sin^2\theta-\theta\sin\theta+18\cos\theta-18)$$

$$+\cos^2\psi_o(\cos^2\theta-\theta\sin\theta-4\cos\theta+3)$$

$$+\sin\psi_o\cos\psi_o(12\theta+\frac{3}{2}\theta\cos\theta-\frac{19}{2}\sin\theta+4\sin\theta\cos\theta)]+0(\alpha^3).$$

Upon substituting equation (100) into equation (105) ρ is obtained in terms of τ, namely,

$$\rho = 1+\alpha[\cos\psi_o(1-\cos\tau)+2\sin\psi_o(\tau-\sin\tau)] \tag{106}$$

$$+\alpha^2[\sin^2\psi_o(3\tau^2+3\tau^2\cos\tau+5\cos^2\tau+6\cos\tau+2\tau\sin\tau-11)$$

$$+\sin\psi_o\cos\psi_o(8\tau+\frac{11}{2}\tau\cos\tau-\frac{3}{2}\tau^2\sin\tau-\frac{19}{2}\sin\tau$$

$$-4\sin\tau\cos\tau)+\sin^2\tau-3\sin\tau-4\cos\tau+4]+0(\alpha^3).$$

We obtain τ in terms of θ from equation (90)

$$\tau = \theta+\alpha[2\cos\psi_o(\theta-\sin\theta)+\sin\psi_o(\frac{3}{2}\theta^2+4\cos\theta-4)] \tag{107}$$

$$+\alpha^2[\sin^2\psi_o(\frac{7}{2}\theta^3+24\sin\theta+6\theta\cos\theta-6\sin\theta\cos\theta-24\theta)$$

$$+\cos^2\psi_o(2\theta\cos\theta+12\sin\theta+\frac{3}{2}\sin\theta\cos\theta+\frac{17}{2}\theta)$$

$$+\sin\psi_o\cos\psi_o(\frac{23}{2}\theta^2+37\cos\theta+\theta\sin\theta-6\cos^2\theta-31)]+(\alpha^3).$$

From equation (107) and using equation (100), θ is obtained in terms of τ

$$\theta = \tau + \alpha[\sin\psi_o(-\frac{3}{2}\tau^2 - 4\cos\tau + 4) - 2\cos\psi_o(\tau - \sin\tau) \tag{108}$$

$$+ \alpha^2[\sin^2\psi_o(\tau^3 - 8\sin\tau + 6\tau\cos\tau - 6\tau^2\sin\tau - 10\sin\tau\cos\tau$$

$$+ 12\tau) + \cos^2\psi_o(8\sin\tau - 6\tau\cos\tau + \frac{5}{2}\sin\tau\cos\tau - \frac{9}{2}\tau)$$

$$+ \sin\psi_o\cos\psi_o(-\frac{5}{2}\tau^2 - 21\cos\tau - 15\tau\sin\tau - 10\cos^2\tau$$

$$- 3\tau^2\cos\tau + 31) + 0(\alpha^3)$$

There is now enough information available to obtain energy and momentum expressions. Using equation (84), (85), and (86), the energy and momentum expressions become

$$E = -\frac{1}{2} + \alpha\tau\sin\psi_o \tag{109}$$

$$+ \alpha^2[\sin^2\psi_o(-\frac{1}{2}\tau^2 - \cos\tau + 1) + \sin\psi_o\cos\psi_o(\tau - \sin\tau$$

$$+ 8\sin\tau\cos\tau) - \cos\tau + 1] + 0(\alpha^3)$$

$$h = 1 + \alpha\tau\sin\psi_o + \alpha^2[\sin^2\psi_o(\tau^2 + 2\cos\tau - 2) \tag{110}$$

$$+ \sin\psi_o\cos\psi_o(\tau - \sin\tau)] + 0(\alpha^3)$$

Using the energy and momentum equations, the following expression is derived for the eccentricity of the instantaneous Kepler ellipse:

$$e^2 = 1 + 2E\,h^2$$

$$e^2 = \alpha^2[6\sin^2\psi_o(1 - \cos\tau) + 16\sin\psi_o\cos\psi_o\sin\tau\cos\tau \tag{111}$$

$$+ 2(1 - \cos\tau)] + 0(\alpha^3)$$

The previously developed equations constitute second-order solutions to equations (54) and (55) for $0 < \alpha << 1$, ψ_o a constant, and θ, τ, and s not too large.

3.1.4 NUMERICAL RESULTS

Figure 9 shows representative low-thrust trajectories produced by the second-order perturbation theory, with a constant thrust acceleration of $\alpha = 0.01$ for four specific thrust angles. The four thrust angle values are shown, and the values of the dimensionless time parameter τ are indicated. The radial thrusting case, $\psi_o = 0^o$, is seen to exhibit the oscillatory behavior described in Section 2.1. Also, the apsidal angle is seen to be approximately $(1 + \alpha)\pi$, which is also apparent from our numerical results. The circumferential thrusting case, $\psi_o = 90^o$, is seen to exhibit the secular increase in radius as predicted in Section 2.3. However, the secular increase shown in Figure 9 is slightly modified by the second-order terms.

The accuracy of the perturbation solutions for arbitrary steering angles cannot be determined from the numerical results shown in Figure 9. Since there are no exact solutions for arbitrary steering angles, we must take a closer look at the underlying assumption upon which our perturbation theory is based. The assumption is that $0 < \alpha << 1$. With this assumption on α, the binomial, exponential, and trigonometric series may be used for convenience in the derivation of the perturbation solutions. When α is sufficiently small, these infinite series converge rather quickly. Therefore, since our perturbation solutions are only infinite series developed through the second order, the accuracy will increase as α becomes smaller. This assumption directly limits the size of the independent variable, either τ or θ, for reasonable accuracy. In general, as α approaches zero, the independent variable can be made larger.

The exact solution for the special case when $\psi_o = 0^o$ is known in terms of elliptic integrals, as noted in Section 2.1. When the elliptic integrals were evaluated and compared with the perturbation theory results, it was found at a given value of ρ at $\tau = \pi$ and $\alpha = 0.01$, the error was less than 1% in τ. However, for $\alpha < 0.001$, the error should be significantly less,

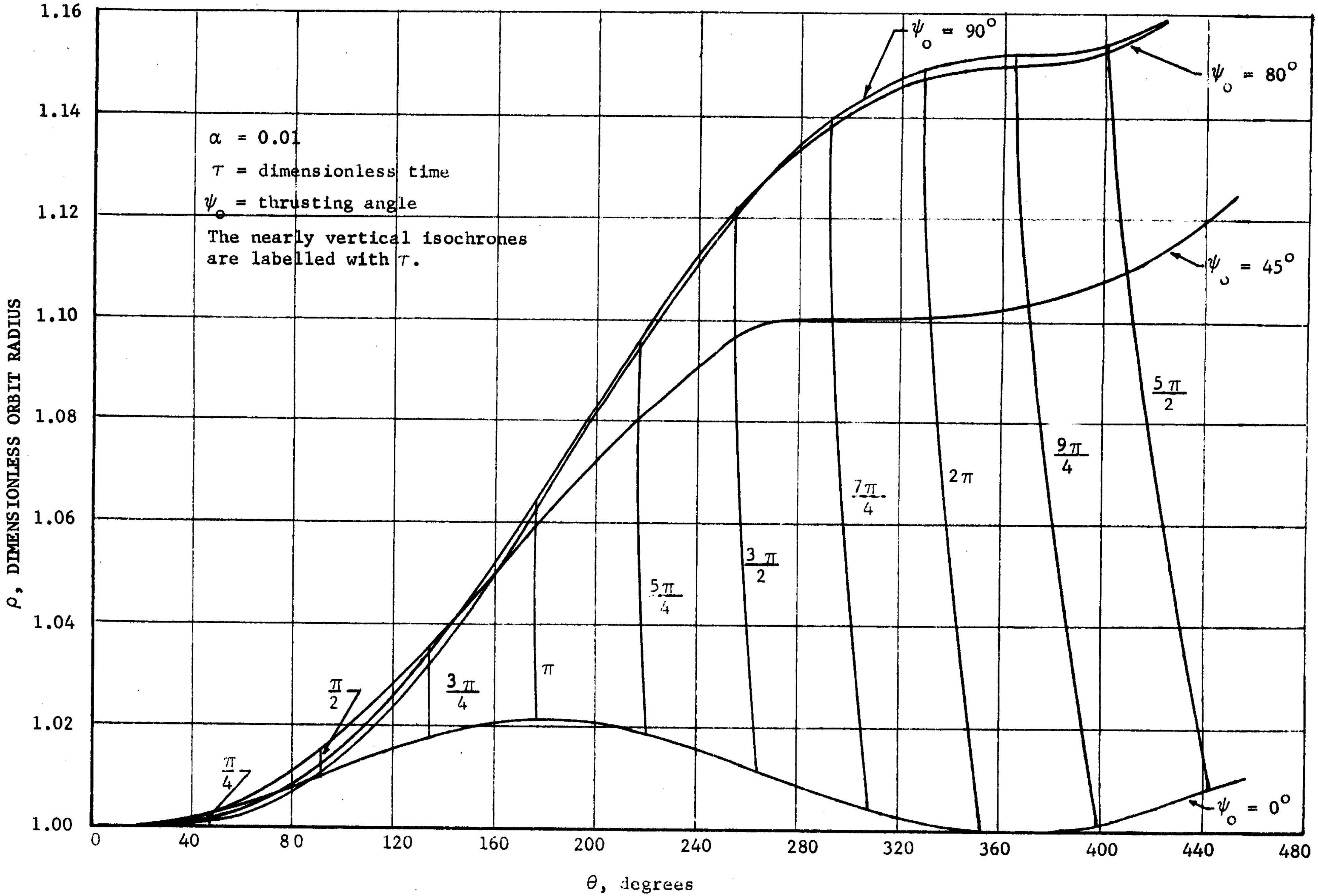


FIGURE 9. REPRESENTATIVE LOW THRUST TRAJECTORIES

and the perturbation solutions would probably give accurate results for one revolution or more. A procedure for improving the accuracy of multiple revolution trjaectories is given in Section 3.2.2.

3.1.5 ORBIT TRANSFER APPLICATION

The perturbation theory may be applied almost directly to the problem of transfer from one circular orbit to another. For transfers involving an initial thrusting phase on departure from a circular orbit followed by a coasting phase and a subsequent thrusting phase to establish the final circular orbit, it is only necessary to choose the thrusting angles and thrust durations so that the energy and momentum values at the beginning and end of the coasting phase are identical. Figure 10 illustrates a sample transfer.

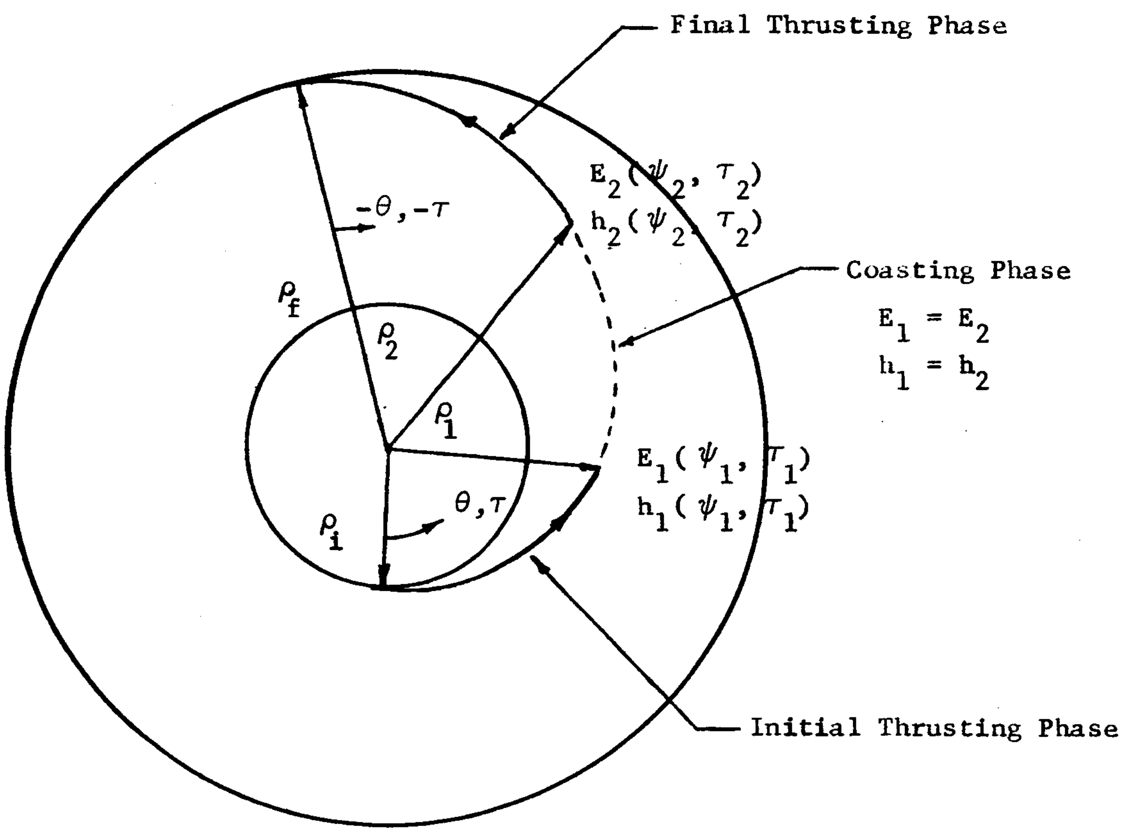


FIGURE 10. ORBIT TRANSFER GEOMETRY

The final boundary condition of the second thrusting phase may be satisfied by interpreting it as an initial condition and then considering the motion in negative time. Because the various equations were non-dimensionalized with regard to the initial orbit, it is necessary to examine the conversion factors between quantities measured in the $\rho_i = 1$ system and the $\rho_f = 1$ system. Denoting the ρ_f quantities by primes, and defining $K = \frac{\rho_f}{\rho_i}$, the conversion factors are:

($\rho_i = 1$ quantity)	x	conv. factor	=	($\rho_f = 1$ quantity)
E		K		E'
h		$K^{-1/2}$		h'
α		K^2		α'
τ		$K^{3/2}$		τ'
ρ (length)		K^{-1}		ρ'(length)
$\dot{\theta}$		$K^{3/2}$		$\dot{\theta}'$
$\dot{\rho}$		$K^{1/2}$		$\dot{\rho}'$

When these conversion factors are utilized, the non-dimensional equations are sufficient to define the transfer maneuver.

3.2 SOME GENERALIZATIONS OF THE PERTURBATION THEORY

3.2.1 EXTENSION TO ELLIPTICAL ORBIT STARTING CONDITIONS

Since the development of the perturbation theory is complete through second-order for circular orbit starting conditions, it is quite natural to extend the theory to accept elliptical orbit starting conditions. To accomplish this without recourse to numerical methods or a completely different analytical approach, the following assumptions were made with regard to the starting conditions:

$$\begin{aligned}
\rho &= \rho_o = 1 + \Delta\rho & \Delta\rho &= 0(\alpha) \\
\dot{\theta} &= \dot{\theta}_o = 1 + \Delta\dot{\theta} & \Delta\dot{\theta} &= 0(\alpha) \\
\dot{\rho} &= \dot{\rho}_o & \dot{\rho}_o &= 0(\alpha) \; .
\end{aligned}$$

These conditions insure that the starting orbit has an eccentricity no larger than order α. The above initial conditions, along with $\theta = \theta_o = 0$ and $\tau = \tau_o = 0$, can be used to re-evaluate the constants of integration required in the perturbation analysis. In view of the similarity of this analysis to that of the preceding subsection, and to avoid undue algebraic complexity, only the first-order result has been derived. The first-order results are as follows:

$$\begin{aligned}
\rho = {} & 1 + \Delta\rho\,(4-3\cos\tau) + 2\,\Delta\dot{\theta}\,(1-\cos\tau) + \dot{\rho}_o \sin\tau \\
& + \alpha[\,2\sin\psi_o\,(\tau - \sin\tau) + \cos\psi_o\,(1-\cos\tau)] + 0(\alpha^2)
\end{aligned} \tag{112}$$

$$\begin{aligned}
\theta = {} & \tau - 6\,\Delta\rho\,(\tau - \sin\tau) - \Delta\dot{\theta}(3\tau - 4\sin\tau) - 2\dot{\rho}_o\,(1-\cos\tau) \\
& - \alpha[\,\sin\psi_o\,(\tfrac{3}{2}\tau^2 + 4\cos\tau - 4) + 2\cos\psi_o\,(\theta - \sin\theta)\,] + 0(\alpha^2)
\end{aligned} \tag{113}$$

$$E = -\frac{1}{2} + 2\,\Delta\rho + \Delta\dot{\theta} + \alpha\tau\sin\psi_o + 0(\alpha^2) \tag{114}$$

$$h = 1 + 2\,\Delta\rho + \Delta\dot{\theta} + \alpha\tau\sin\psi_o + 0(\alpha^2) \; . \tag{115}$$

By equation (113), $\tau = \theta + 0(\alpha)$, which implies that equations (112), (114), and (115) can be written in terms of θ by just substituting θ for τ.

3.2.2 TRAJECTORIES INVOLVING MULTIPLE REVOLUTIONS

In a so-called strong gravitational field (thrust acceleration extremely small compared to gravitational acceleration) trajectories of interest may involve several revolutions about the central body. Direct application of the perturbation theory of subsection 3.1 would result in significant error accumulation for such long transfer times. A more sensible approach would be to re-establish the circular reference orbit at each revolution. This technique, having an analog in the rectification process in the Encke method of special perturbation theory, should sharply reduce the overall error accumulation. It will be shown that by appropriately choosing the time of rectification, the first-order extension to elliptical orbit starting conditions provides the information necessary for performing a rectification which is consistent to the second-order. Differentiating equations (112) and (113) gives

$$\dot{\rho} = 3\Delta\rho \sin\tau + 2\Delta\dot{\theta}\sin\tau + \dot{\rho}_o \cos\tau + \alpha[2\sin\psi_o(1-\cos\tau) + \cos\psi_o \sin\tau] + 0(\alpha^2), \tag{116}$$

$$\dot{\theta} = 1 - 6\Delta\rho(1-\cos\tau) - \Delta\dot{\theta}(3-4\cos\tau) + 2\dot{\rho}_o \sin\tau - \alpha[\sin\psi_o(3\tau - 4\sin\tau) + 2\cos\psi_o(1-\cos\tau)] + 0(\alpha^2). \tag{117}$$

Assume that the initial starting conditions are such that $\Delta\rho$, $\Delta\dot{\theta}$, and $\dot{\rho}_o$ are of order α^2 or smaller. Next consider equations (116) and (117) for $\tau = 2\pi$:

$$\dot{\rho}(2\pi) = 0 + 0(\alpha^2), \tag{118}$$

$$\dot{\theta}(2\pi) = 1 - \alpha\, 6\pi \sin\psi_o + 0(\alpha^2). \tag{119}$$

From equation (112)

$$\rho(2\pi) = 1 + \alpha\, 4\pi \sin\psi_o + 0(\alpha^2). \tag{120}$$

Next compare $\dot{\theta}(2\pi)$ with the circular orbit value $\dot{\theta}_{co}$ for $\rho = \rho(2\pi)$.

$$\dot{\theta}_{co} = \frac{1}{[\rho(2\pi)]^{3/2}} = \frac{1}{[1 + \alpha 4\pi \sin\psi_o + (\alpha^2)]^{3/2}}$$

$$\dot{\theta}_{co} = 1 - \alpha 6\pi \sin\psi_o + 0(\alpha^2) \tag{121}$$

From equations (118), (119), and (120) it is seen that conditions at $\tau = 2\pi$ are those of a circular orbit to order α. Since the deviations from a circular orbit condition are of order α^2, equations (112), (113), (114) and (115), along with the terms of order α^2 from subsection 3.1, are sufficient for the multiple revolution case. Higher order terms in $\Delta\dot{\theta}$ and $\dot{\rho}_o$ are not required if the rectification occurs at $\tau = 2\pi$, since the coefficients of these terms are already of order α^2.

A step by step description of a multiple revolution trajectory computation follows:

(1) Using equations (112), (113), (114) and (115) with second-order terms included, compute the time histories to $\tau = 2\pi$. Remember that $\Delta\rho$, $\dot{\theta}_o$, and $\dot{\rho}_o$ must initially be of order α^2.

(2) Using the conversion factors contained in 3.1.5, convert the quantities to a system with a reference orbit of radius $\rho = \rho(2\pi)$. That is,

$$\tau, \rho(2\pi), \dot{\rho}(2\pi), \dot{\theta}(2\pi) \longrightarrow \tau', 1, \dot{\rho}'(0), \dot{\theta}'(0).$$

(3) Establish the initial deviations from the new reference orbit.

$$\Delta\dot{\theta}' = \dot{\theta}'(0) - 1$$

$$\dot{\rho}'_o = \dot{\rho}'(0)$$

(4) Again using equations (112), (113), (114), and (115), compute the time histories to $\tau' = 2\pi$. For output purposes, it may be desirable to reconvert the results to the original system of units.

(5) Repeat the above steps until the desired terminal condition is reached.

The accuracy obtained using this technique will depend mainly upon the value of α employed. For extremely small α, it may be satisfactory to defer rectification to $\tau = 2n\pi$ where $n > 1$. It is again emphasized that the above procedure is entirely consistent to order α^2.

3.2.3 SOLUTIONS WITH VARIABLE THRUST ACCELERATION

Using a procedure similar to that of Section 3.2.1, it is possible to account for time variations in the thrust acceleration parameter α. While the solution could easily be obtained for any sort of time variation, the problem of most significance involves a linear change in α with time. This would correspond to a vehicle with constant thrust whose mass decreases at a constant rate. Let α of the preceding subsections be defined as α_o, then consider α to be given by $\alpha = \alpha_o + \dot{\alpha}\tau$, where $\dot{\alpha}$ is sufficiently small so that $\dot{\alpha}\tau$ is no larger than order α_o. Let $\delta\rho(\tau)$, $\delta\dot{\rho}(\tau)$, $\delta\theta(\tau)$, and $\delta\dot{\theta}(\tau)$ define the change in position and velocity resulting from the time variant portion of the thrust acceleration. First-order perturbation solutions for these quantities are

$$\delta\rho = \dot{\alpha}\,[\cos\psi_o(\tau - \sin\tau) + \sin\psi_o(\tau^2 + 2\cos\tau - 2)] + 0(\alpha_o^2), \quad (122)$$

$$\delta\dot{\rho} = \dot{\alpha}\,[\sin\psi_o(1 - \cos\tau) + \sin\psi_o(2\tau - 2\sin\tau)] + 0(\alpha_o^2), \quad (123)$$

$$\delta\theta = \dot{\alpha}\,[\sin\psi_o(4\tau - 4\sin\tau - \frac{\tau^3}{2}) + \cos\psi_o(2 - 2\cos\tau - \tau^2)] + 0(\alpha_o^2), \quad (124)$$

$$\delta\dot{\theta} = \dot{\alpha}[\sin\psi_o(4 - 4\cos\tau - \frac{3\tau}{2}) + \cos\psi_o(2\sin\tau - 2\tau)] + 0(\alpha_o^2). \quad (125)$$

These perturbative terms, when added to the previously derived first-order solutions for the α_o contribution (subsections 3.1.3, 3.2.1, and 3.2.2) yield a first-order solution for the linear time-varying thrust acceleration case. It should be pointed out that if $\dot{\alpha}$ is very small so that $\dot{\alpha}\tau$ is of order α_o^2 for τ values of interest, the above equations represent a second-order contribution, and may be used in conjunction with the previously derived second-order theory (subsection 3.1.3).

SECTION 4

CONCLUSIONS

The preceding investigations have considered the in-plane trajectories resulting from the application of a small thrust to an object in an inverse square central force field.

The analytical solutions for thrust programs involving radial, normal, circumferential, and tangential thrusting were reviewed and extended. For the radial thrusting case, it was found that when $|\alpha| << 1/8$, the radius vector is periodic between r_o and $(1 + 2\alpha)r_o$. The frequency is $(1 + 3\alpha)n_o$ and apsidal angle is $(1 + \alpha)\pi$, where n_o is the mean motion of the initial circular orbit.

For the normal thrusting case, the totality of motions for a particle initially in a circular Kepler orbit was determined. It was found that the orbits lie in a ring bounded by two circles, the first with radius equal to the radius of the initial Kepler orbit and the second with radius dependent on the normal force. The second circle lies outside the first circle when the normal force is outward and lies inside when the normal force is inward. The radius of the second circle cannot exceed twice the radius of the first circle and is reached only when the normal force is 0.230 times the gravity force at the initial radius. The point of central attraction is reached only when the normal force is 2.809 times the gravity force of the initial radius. The orbit path oscillates periodically between the two circles. However, the orbits are not in general periodic since they do not close. When the magnitude of the normal force is small, the orbits are direct, while when the force is large, the orbits are direct near the first circle and retrograde near the second circle.

For the tangential and circumferential thrust case, the complete first-order solutions were derived using the asymptotic method due to Kryloff and Bogoliuboff. The similarity of the derived first-order expressions for the radius vector to those determined by Benney, using the perturbation approach, indicates that Benney's monotonic term is a secular term rather than the initial term of an infinite series representing a periodic function. This distinction is not evident from a perturbation analysis.

The investigation has also yielded perturbation solutions of the differential equations of motion of a vehicle moving under low thrust. Second-order solutions were derived for a vehicle departing from a circular orbit. Also first-order solutions are given for a vehicle departing from an elliptical orbit with small eccentricity. The thrust vector is assumed to form an arbitrary, but constant, angle with the radius vector. The solutions are accurate when the ratio of the thrust acceleration to the initial gravitational acceleration is much less than unity and the time or the polar angle measured from the initial position is not too large. Trajectories involving several revolutions would have less error accumulation if the reference orbit were re-established after each revolution. The necessary analysis was carried out, using the first-order perturbation theory for elliptical orbit starting conditions, to accomplish the re-establishment of the reference orbit. This technique gives a solution consistent through second-order. Since these perturbation solutions are applicable to arbitrary thrust angles, they complete a class of trajectories which includes previously published solutions for radial and circumferential thrusting.

Further analysis is required to establish the accuracy of the perturbation solutions as a function of thrust acceleration magnitude and time. This could be accomplished by comparisons with precision numerically computed trajectories. Additional effort could be devoted to application of the results. A whole class of optimzation problems, namely those involving the selection of an optimal thrust direction, can be treated using the perturbation solutions. These results would find application to spacecraft in which the thrust vector, for guidance reasons, is most conveniently directed at a constant angle to the radius vector. Other obvious extensions of the perturbation results would include the effect of time varying thrust angles and larger starting orbit eccentricities. Provision for treating out-of-plane motion would also be desirable.

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